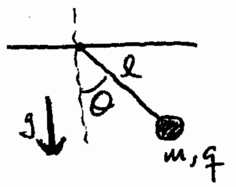
Qual Review, Meeting 6: Potpourri 2

1. A narrow beam of length L and mass per unit length ρ has both ends firmly clamped so that their displacements and spatial derivatives vanish. When the beam is bent, the total energy stored in the beam (neglecting gravity) is equal to $U = \frac{k}{2}\int_0^L \left(\frac{\partial^2 s}{\partial x^2}\right)^2 dx$, where $s(x)$ is the transverse displacement (assumed to be small) and k is a constant.

1a. Find the differential equation satisfied by $s(x,t)$ for small displacements.

1b. Find the eigenfrequencies ν_n. Give the transcendental equation of which the ν_n are solutions. Give an explicit formula for ν_n in the limit of large n.



2. A mass m is suspended by a rod of length l. The mass carries charge q and is in a uniform magnetic field B and a uniform gravitational field g, both pointing downward. The rod is completely rigid, and can assume any position (θ, ϕ) as long as $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$. If the mass is released from rest at $\theta = \frac{\pi}{2}$, what is the minimum value of θ reached in the subsequent motion? You may assume $\Omega^2 l << g$, where $\Omega = \frac{qB}{mc}$.

3. Consider a one-dimensional simple harmonic oscillator whose classical frequency is ω_0. Let $|n\rangle$ be the nth excited state of the oscillator.

3a. Derive expressions for $\langle n'|x|n\rangle$ and $\langle n'|p|n\rangle$.

3b. Suppose that for $t < 0$ the oscillator is in the ground state. At $t = 0$ the oscillator is subjected to a spatially uniform harmonically varying force $-F_0 cos(\omega t)$, with $\omega \neq \omega_0$. Obtain an expression for the expectation value $\langle x\rangle$ as a function of time to lowest nonvanishing order.

4. The Einstein model of lattice vibrations of a solid consisting of N atoms represents the solid by $3N$ harmonic oscillators, each with frequency ω_0. Answer the following questions about this model.

4a. Find the mean energy of the system as a function of temperature.

4b. Find the heat capacity of the system, and evaluate it in the high temperature limit. Discuss this result in terms of the equipartition theorem.

4c. To model anharmonic effects, one assumes that the frequency ω_0 is a function of the volume V. Assuming this to be true, find the pressure as a function of $\left(\frac{\partial \omega_0}{\partial V}\right)_T$.

Potpourri 2 Solns

1) $U = \frac{K}{2}\int_0^L \left(\frac{\partial^2 s}{\partial x^2}\right)^2 dx.$

Writing $S = \int L\,dt = \int \mathcal{L}\,dx\,dt$, use principle of least action,

with $\mathcal{L} = \mathcal{L}\left(\dot{s}, \frac{\partial^2 s}{\partial x^2}\right)$

$$\Rightarrow S = \int \mathcal{L}\left(\dot{s}, \frac{\partial^2 s}{\partial x^2}\right) dx\,dt \Rightarrow \delta S = \int\left[\frac{\partial \mathcal{L}}{\partial \dot{s}}\delta\dot{s} + \frac{\partial \mathcal{L}}{\partial(\partial_x^2 s)}\delta(\partial_x^2 s)\right] dx\,dt$$

$$\Rightarrow \delta S = \int\left[-\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x^2 s)}\right)\right]\delta s\,dx\,dt$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial(\partial_x^2 s)}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{s}}\right) = 0}$$

Here, $\mathcal{L} = \frac{1}{2}\rho\left(\frac{\partial s}{\partial t}\right)^2 - \frac{K}{2}\left(\frac{\partial^2 s}{\partial x^2}\right)^2$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{s}} = \rho\frac{\partial s}{\partial t}; \quad \frac{\partial \mathcal{L}}{\partial(\partial_x^2 s)} = -K\left(\frac{\partial^2 s}{\partial x^2}\right)$$

$$\Rightarrow -K\left(\frac{\partial^4 s}{\partial x^4}\right) - \rho\frac{\partial^2 s}{\partial t^2} = 0 \Rightarrow \boxed{\frac{\partial^4 s}{\partial x^4} = -\frac{\rho}{K}\frac{\partial^2 s}{\partial t^2}}$$

2] Clearly, try Fourier modes: $s = S_0 e^{i(qx-\omega t)}$

$\Rightarrow q^4 = \frac{\rho}{k}\omega^2$. Normally, we'd have $q = \pm(\ )$. But here, it's $\underline{q^4}$, so can have $q = \pm(\ ), \pm i(\ )$.

$\Rightarrow$ Can in general have sin, cos, sinh, cosh sols!

Possibly phase.

$$\Rightarrow s = \left[A\sin(qx) + B\cos(qx) + C\sinh(qx) + D\cosh(qx)\right]e^{-i(\omega t+\phi)}$$

B.C.'s: $s(0,t) = 0$, $s'(0,t) = 0$

$s(L,t) = 0$, $s'(L,t) = 0$.

$$\Rightarrow B + D = 0$$
$$A + C = 0$$
$$A\sin(qL) + B\cos(qL) + C\sinh(qL) + D\cosh(qL) = 0$$
$$A\cos(qL) - B\sin(qL) + C\cosh(qL) + D\sinh(qL) = 0$$

$$\Rightarrow A(\sin(qL) - \sinh(qL)) + B(\cos(qL) - \cosh(qL)) = 0$$
$$A(\cos(qL) - \cosh(qL)) - B(\sin(qL) + \sinh(qL)) = 0$$

$$\Rightarrow -(\sin(qL) - \sinh(qL))(\sin(qL) + \sinh(qL)) - (\cos(qL) - \cosh(qL))^2 = 0$$

$$\Rightarrow \sin^2 qL + \cos^2 qL + \cosh^2 qL - \sinh^2 qL - 2\cos(qL)\cosh(qL) = 0$$

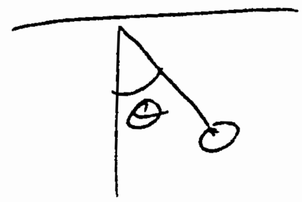
$$\Rightarrow \boxed{\cos(qL)\cosh(qL) = 1}$$

large $n \to$ lots of oscillations $\Rightarrow qL >> 1$.

$\cosh(qL) >> 1 \Rightarrow \cos(qL) \approx 0.$

$$qL = \frac{2n+1}{2}\pi$$

$$\to \boxed{q = \left(n+\frac{1}{2}\right)\frac{\pi}{L}}$$



2] $L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \ell^2 \sin^2\theta\,\dot{\phi}^2) + \frac{e}{c}\underset{\sim}{v}\cdot\underset{\sim}{A} - mgz.$

Here, $\underset{\sim}{B} = -B_0\hat{z}$, so $\oint \underset{\sim}{A}\cdot d\ell = -B_0 \pi r_\perp^2 \Rightarrow \underset{\sim}{A} = \hat{\phi}\left(-\frac{B_0 r_\perp}{2}\right)$

So $\frac{e}{c}\underset{\sim}{v}\cdot\underset{\sim}{A} = \frac{e}{c}v_\phi A_\phi = \frac{e}{c}r_\perp\dot{\phi}\left(\frac{-B_0 r_\perp}{2}\right) = -\frac{eB_0}{2c}\ell^2\sin^2\theta\,\dot{\phi}$

$z = -\ell\cos\theta$,

$$\Rightarrow \boxed{L = \frac{1}{2}m(\ell^2\dot{\theta}^2 + \ell^2\sin^2\theta\,\dot{\phi}^2) - \frac{eB_0}{2c}\ell^2\sin^2\theta\,\dot{\phi} + mg\ell\cos\theta}$$

$$\frac{\partial L}{\partial\dot{\phi}} = p_\phi = m\ell^2\sin^2\theta\,\dot{\phi} - \frac{eB_0}{2c}\ell^2\sin^2\theta = \text{const.}$$

Initially, $\theta = \frac{\pi}{2}$, $\dot{\phi} = 0$, so $p_\phi = \frac{-eB_0\ell^2}{2c}$

$$\Rightarrow \dot{\phi} = \frac{-(eB_0\ell^2/2c)(1-\sin^2\theta)}{m\ell^2\sin^2\theta} = -\frac{eB}{2mc}\frac{\cos^2\theta}{\sin^2\theta} = -\frac{\Omega}{2\tan^2\theta}.$$

At all times, $\mathcal{H} = E = \text{const.}$

$$H = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi}^2 - mgl\cos\theta \quad (\underline{A} \text{ drops out})$$

at ~~start~~ beginning, $\dot{\theta} = \dot{\phi} = \cos\left(\frac{\pi}{2}\right) = 0. \Rightarrow E = 0.$

at min. θ, $\dot{\theta} = 0$

$$\Rightarrow \frac{1}{2}ml^2\sin^2\theta\,\dot{\phi}^2 = mgl\cos\theta$$

$$\Rightarrow \frac{1}{2}\cancel{m}l^2\sin^2\theta\,\frac{\Omega^2\cos^4\theta}{4\sin^4\theta} = \cancel{m}g\cancel{l}\cos\theta$$

$$\Rightarrow \sin^2\theta = \frac{\Omega^2 l}{8g}\cos^2\theta$$

For small θ, $\boxed{\theta_{min} \simeq \sqrt{\frac{\Omega^2 l}{8g}}}$

3a] $a=\sqrt{\frac{m\omega_0}{2\hbar}}\left(x+\frac{ip}{m\omega_0}\right)$, $a^+=\sqrt{\frac{m\omega_0}{2\hbar}}\left(x-\frac{ip}{m\omega_0}\right)$

$$\Rightarrow x=\sqrt{\frac{\hbar}{2m\omega_0}}\,(a+a^+),\quad p=i\sqrt{\frac{m\hbar\omega_0}{2}}\,(a-a^+)$$

Using $\langle n'|a|n\rangle=\sqrt{n}\,\delta_{n',n-1}$ & $\langle n'|a^+|n\rangle=\sqrt{n+1}\,\delta_{n',n+1}$

We get $\langle n'|x|n\rangle=\sqrt{\frac{\hbar}{2m\omega_0}}\left(\sqrt{n}\,\delta_{n',n-1}+\sqrt{n+1}\,\delta_{n',n+1}\right)$

$$\langle n'|p|n\rangle=i\sqrt{\frac{m\hbar\omega_0}{2}}\left(-\sqrt{n}\,\delta_{n',n-1}+\sqrt{n+1}\,\delta_{n',n+1}\right).$$

b] $V(t)=F_0\,x\cos(\omega t)$

$$C_n^1=\frac{-i}{\hbar}\int_0^t e^{i\omega_{ni}t'}F_0\langle n|x|i\rangle\left(\frac{e^{i\omega t'}+e^{-i\omega t'}}{2}\right)dt'$$

$$=\frac{-iF_0}{2\hbar}\langle n|x|i\rangle\int_0^t\left(e^{i(\omega_{ni}+\omega)t'}+e^{i(\omega_{ni}-\omega)t'}\right)dt'$$

$$=\frac{-iF_0}{2\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\sqrt{1}\,\delta_{n,1}\left[\frac{e^{i(\omega_{ni}+\omega)t}-1}{i(\omega_{ni}+\omega)}+\frac{e^{i(\omega_{ni}-\omega)t}-1}{i(\omega_{ni}-\omega)}\right]$$

$\Rightarrow$ Only nonvanishing C is for $|1\rangle$:

$\omega_{10}=\omega_0$

Remember, $|\psi(t)\rangle=e^{-iH_0t/\hbar}|\varphi_I\rangle$
$=e^{-iH_0t/\hbar}\sum c_n|$

So $\langle x\rangle=\langle\psi|x|\psi\rangle$, where

$-iHt/\hbar$

$$\Rightarrow |\psi\rangle = e^{-i\omega_0 t/2}|0\rangle + c_1 e^{-3i\omega_0 t/2}|1\rangle$$

$$\Rightarrow \langle x\rangle = \left(\langle 0| e^{i\omega_0 t/2} + c_1^* e^{3i\omega_0 t/2}\langle 1|\right) x \left(e^{-i\omega_0 t/2}|0\rangle + c_1 e^{-3i\omega_0 t/2}|1\rangle\right)$$

only the off-diagonal elts. survive $\rightarrow$

$$\langle x\rangle = c_1 e^{-i\omega_0 t}\langle 0|x|1\rangle + c_1^* e^{i\omega_0 t}\langle 1|x|0\rangle.$$

$$= \frac{-F_0}{2\hbar}\sqrt{\frac{\hbar}{2m\omega_0}}\left[\frac{e^{i\omega_0 t}e^{i\omega t}-1}{(\omega_0+\omega)} + \frac{e^{i(\omega_0 t)}e^{-i\omega t}-1}{\omega_0-\omega}\right]e^{-i\omega_0 t}\sqrt{\frac{\hbar}{2m\omega_0}} + \text{c.c.}$$

$$= \frac{-F}{2\hbar}\frac{\hbar}{2m\omega_0}\left[\frac{e^{i\omega t}-e^{-i\omega_0 t}}{\omega_0+\omega} + \frac{e^{-i\omega t}-e^{-i\omega_0 t}}{\omega_0-\omega} + \frac{e^{-i\omega t}-e^{i\omega_0 t}}{\omega_0+\omega} + \frac{e^{i\omega t}-e^{i\omega_0 t}}{\omega_0-\omega}\right]$$

$$= \frac{-F}{4m\omega_0}\left[\frac{2\cos(\omega t)}{\omega_0+\omega} - \frac{2\cos(\omega_0 t)}{\omega_0+\omega} + \frac{2\cos(\omega t)}{\omega_0-\omega} - \frac{2\cos(\omega_0 t)}{\omega_0-\omega}\right]$$

$$= \frac{-F}{2m\omega_0}\left[\frac{\cos(\omega t)\cdot 2\omega_0}{\omega_0^2-\omega^2} + \frac{\cos(\omega_0 t)\cdot 2\omega_0}{\omega_0^2-\omega^2}\right]$$

$$= \boxed{\frac{-F}{m}\left[\frac{\cos(\omega t)-\cos(\omega_0 t)}{\omega_0^2-\omega^2}\right]}$$

$$Z = \left(\sum_n e^{-\beta\hbar\omega(n-\frac{1}{2})}\right)^{3N} = \left(e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}\right)^{-3N}$$

$$\Rightarrow E = -\frac{\partial \ln Z}{\partial \beta} = 3N \frac{\partial}{\partial \beta} \ln\left(e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}\right)$$

$$= 3N \, \frac{\frac{\hbar\omega}{2} e^{\beta\hbar\omega/2} + \frac{\hbar\omega}{2} e^{-\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}$$

$$= \boxed{\frac{3N\hbar\omega}{2}\left[\frac{e^{\beta\hbar\omega/2} + e^{-\beta\hbar\omega/2}}{e^{\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}\right]}$$

or $$= \frac{3N\hbar\omega}{2}\left[\frac{1 + e^{\beta\hbar\omega}}{e^{\beta\hbar\omega} - 1}\right]$$

or $$= 3N\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}\right]$$ etc.

$$C = \frac{\partial E}{\partial T} = 3Nk_B\left(\frac{\hbar\omega}{k_BT}\right)^2 \frac{e^{-\beta\hbar\omega}}{\left(e^{\beta\hbar\omega} - 1\right)^2}$$

High T: $C \simeq 3Nk_B$.

Since each 1D Oscillator has 2 d.o.f., $\varepsilon = k_BT$

$\Rightarrow E \simeq 3Nk_BT$ ✓.

c] $dF = -S\,dT - p\,dV$

$$\Rightarrow \left(\frac{\partial F}{\partial V}\right)_T = -P$$

$F = -k_B T \ln Z$, so

$$p = 3N\hbar\left(\frac{\partial \omega_0}{\partial V}\right)_T \left[\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega_0} - 1}\right].$$