Uncertainty, Measurement, and Models
Overview Exp #1

Lecture # 2
Physics 2BL
Winter 2011
# Lab TAs

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<tr>
<th>Time</th>
<th>Tuesday</th>
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- Heemin: heemin@ucsd.edu
- Stefan: sprogova@ucsd.edu
- Lilana: lil005@ucsd.edu
- Matthew: mmui@ucsd.edu

MYR-A 2722
TA Coordinator

Ben Heldt

bheldt@ucsd.edu

Present in labs to assist
Outline

• What uncertainty (error) analysis can for you
• Issues with measurement and observation
• What does a model do?
• General error propagation formula with example
• Overview of Experiment # 1
• Homework
What is uncertainty (error)?

• Uncertainty (or error) in a measurement is not the same as a mistake
• Uncertainty results from:
  – Limits of instruments
    • finite spacing of markings on ruler
  – Design of measurement
    • using stopwatch instead of photogate
  – Less-well defined quantities
    • composition of materials
Understanding uncertainty is important

- for comparing values
- for distinguishing between models
- for designing to specifications/planning

Measurements are less useful (often useless) without a statement of their uncertainty
An example

Batteries

rated for 1.5 V potential difference across terminals

in reality…
Utility of uncertainty analysis

• Evaluating uncertainty in a measurement
• Propagating errors – ability to extend results through calculations or to other measurements
• Analyzing a distribution of values
• Quantifying relationships between measured values
Evaluating error in measurements

- To measure height of building, drop rock and measure time to fall: \( d = \frac{1}{2} gt^2 \)
- Measure times 2.6s, 2.4s, 2.5s, 2.4s, 2.3s, 2.9s
- What is the “best” value
- How certain are we of it?
Calculate “best” value of the time

• Calculate average value (2.6s, 2.4s, 2.5s, 2.4s, 2.3s, 2.9s)

\[ t = \frac{\sum_{i=1}^{n} t_i}{n} \]

\[ t = 2.51666666666666666666666 \text{ s} \]

• Is this reasonable?

  Significant figures
Uncertainty in time

• Measured values - (2.6s, 2.4s, 2.5s, 2.4s, 2.3s, 2.9s)

• By inspection can say uncertainty < 0.4 s

• Calculate standard deviation

\[ \sigma = \sqrt{\frac{\sum (t_i - \bar{t})^2}{n-1}} \]
\[ \sigma = 0.2137288 \text{ s} \]
\[ \sigma = 0.2 \text{ s} \] (But what does this mean???)
How to quote best value

- What is uncertainty in average value?
  - Introduce standard deviation of the mean
    \[ \sigma_{\bar{t}} = \frac{\sigma}{\sqrt{n}} = 0.08725 \quad s = 0.09 \ s \]
- Now what is best quote of average value
  - \( \bar{t} = 2.51666666666666666666666 \ s \)
  - \( \bar{t} = 2.52 \ s \)
- Best value is
  - \( \bar{t} = 2.52 \pm 0.09 \ s \)
Propagation of error

• Same experiment, continued…
• From best estimate of time, get best estimate of distance: 31 meters
• Know uncertainty in time, what about uncertainty in distance?
• From error analysis tells us how errors propagate through mathematical functions
  (2 meters)
Expected uncertainty in a calculated sum \( a = b + c \)

- Each value has an uncertainty
  - \( b = \bar{b} \pm \delta b \)
  - \( c = \bar{c} \pm \delta c \)

- Uncertainty for \( a \) (\( \delta a \)) is \textbf{at most} the sum of the uncertainties
  \( \delta a = \delta b + \delta c \)

- Better value for \( \delta a \) is
  \( \delta a = \sqrt{(\delta b^2 + \delta c^2)} \)

- Best value is
  - \( a = \bar{a} \pm \delta a \)
Expected uncertainty in a calculated product $a = b \times c$

- Each value has an uncertainty
  - $b = b \pm \delta b$
  - $c = c \pm \delta c$
- Relative uncertainty for $a$ ($\varepsilon a$) is at most the sum of the RELATIVE uncertainties
  $$\varepsilon a = \frac{\delta a}{a} = \varepsilon b + \varepsilon c$$
- Better value for $\delta a$ is
  $$\varepsilon a = \sqrt{(\varepsilon b^2 + \varepsilon c^2)}$$
- Best value is
  - $a = a \pm \varepsilon a$ (fractional uncertainty)
What about powers in a product

\[ a = b \cdot c^2 \]

– Each value has an uncertainty
  • \( b = b \pm \delta b \)
  • \( c = c \pm \delta c \)
  • \( \varepsilon a = \frac{\delta a}{a} \) (relative uncertainty)
– powers become a prefactor (weighting) in the error propagation
  • \( \varepsilon a^2 = (\varepsilon b^2 + (2 \varepsilon c)^2) \)
How does uncertainty in $t$ effect the calculated parameter $d$?

\[ d = \frac{1}{2} g t^2 \]

\[ \varepsilon d = \sqrt{(2 \cdot \varepsilon t)^2} = 2 \cdot \varepsilon t \]

\[ \varepsilon d = 2 \cdot \frac{0.09}{2.52} = 0.071 \]

\[ \delta d = 0.071 \times 31 \text{ m} = 2.2 \text{ m} = 2 \text{ m} \]

Statistical error
Relationships

• Know there is a functional relation between $d$ and $t$ $d = \frac{1}{2} g t^2$
• $d$ is directly proportional to $t^2$
• Related through a constant $\frac{1}{2} g$
• Can measure time of drop ($t$) at different heights ($d$)
• plot $d$ versus $t$ to obtain constant
Quantifying relationships

\[ d = \frac{1}{2} gt^2 \]

**FIT:**
\[ g = 8.3 \pm 0.3 \text{ m/s}^2 \]

**d = \frac{1}{2} g(t^2)**

**Fit:**
\[ \text{slope} = 4.3 \pm 0.2 \text{ m/s}^2 \]
\[ \text{intercept} = -10 \pm 10 \text{ m} \]

\[ g = 8.6 \pm 0.4 \text{ m/s}^2 \]
General Formula for error propagation

For independent, random errors

\[ \delta q = \left| \frac{dq}{dx} \right| \delta x \]

\[ \delta q = \sqrt{\left( \frac{\partial q}{\partial x} \delta x \right)^2 + \left( \frac{\partial q}{\partial y} \delta y \right)^2} \]
Measurement and Observation

- Measurement: deciding the amount of a given property by observation
- Empirical
- Not logical deduction
- Not all measurements are created equal...
Reproducibility

• Same results under similar circumstances
  – Reliable/precise

• ‘Similar’ - a slippery thing
  – Measure resistance of metal
    • need same sample purity for repeatable measurement
    • need same people in room?
    • same potential difference?
  – Measure outcome of treatment on patients
    • Can’t repeat on same patient
    • Patients not the same
Precision and Accuracy

• Precise - reproducible
• Accurate - close to true value
• Example - temperature measurement
  – thermometer with
    • fine divisions
    • or with coarse divisions
  – and that reads
    • 0 °C in ice water
    • or 5 °C in ice water
Accuracy vs. Precision
Random and Systematic Errors

- Accuracy and precision are related to types of errors
  - random (thermometer with coarse scale)
    - can be reduced with repeated measurements, careful design
  - systematic (calibration error)
    - difficult to detect with error analysis
    - compare to independent measurement
Observations in Practice

- Does a measurement measure what you think it does? Validity
- Are scope of observations appropriate?
  - Incidental circumstances
  - Sample selection bias
- Depends on model
Models

• Model is a construction that represents a subject or imitates a system
• Used to predict other behaviors (extrapolation)
• Provides context for measurements and design of experiments
  – guide to features of significance during observation
Testing model

- Models must be consistent with data
- Decide between competing models
  - elaboration: extend model to region of disagreement
  - precision: prefer model that is more precise
  - simplicity: Ockham’s razor
Experiment 1 Overview: Density of Earth

\[ \rho = \frac{\frac{M_E}{\frac{4}{3} \pi R_E^3}}{\frac{3g}{4\pi GR_E^2}} = \frac{GM_Em}{R_E^2} = mg \]

\[ R_E = \frac{2h}{\omega^2(\Delta t)^2} \]

measure \( \Delta t \) between sunset on cliff and at sea level
Experiment 1: Height of Cliff

- Range finder to get $L$
- Sextant to get $\theta$
- Wear comfortable shoes
- Make sure you use $\theta$ and not $(90 - \theta)$
Measure Earth’s Radius using $\Delta t$ Sunset

Now, is this time delay measurable?

$$t = \frac{L}{2\pi R_e} \quad T = \frac{T}{2\pi} \sqrt{\frac{2h}{R_e}}$$

$T = 24 \text{ hr} = 24 \cdot 60 \cdot 60 \text{ s}$

$= 86400 \text{ s}$

$R_e = 6,000,000 \text{ m}$

$h \sim 100 \text{ m}$ - our cliff

$$t = \frac{86400 \text{ s}}{2\pi} \sqrt{\frac{200}{6 \times 10^6}} \approx 80 \text{ s}$$

Looks doable!

$h$ - height above the sea level

$L$ - distance to the horizon line

Sunset on earth’s surface (tangent)

Sunset at height $h$

Have we forgotten something?
"The Equation" for Experiment 1a

\[ t = \frac{T}{2\pi} \sqrt{\frac{2Ch}{R_e}} = \frac{1}{\omega} \sqrt{\frac{2Ch}{R_e}} \]

from previous page.

\[ \Delta t = t_1 - t_2 = \frac{1}{\omega} \sqrt{\frac{2C}{R_e}} (\sqrt{h_1} - \sqrt{h_2}) \]

\[ C \equiv \frac{1}{\cos^2(\lambda) \cos^2(\lambda_s) - \sin^2(\lambda) \sin^2(\lambda_s)} \]

Which are the variables that contribute to the error significantly?

Time difference between the two sunset observers.

Season dependant factor slightly greater than 1.

The formula for your error analysis.

\[ R_e = \frac{2C}{\omega^2} \left( \frac{\sqrt{h_1} - \sqrt{h_2}}{\Delta t} \right)^2 \]

What other methods could we use to measure the radius of the earth?

Eratosthenes

angular deviation = angle subtended

From Yagil
Experiment 1:
Determine g

pendulum

\[ F = -mg \sin(\phi) = -mg\phi \]

\[ F = m\alpha = ml\ddot{\phi} \]

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}} \]
Experiment 1: Pendulum

- For release angle $\theta_i$, you should have a set of time data $(t^p_1, t^p_2, t^p_3, \ldots, t^p_N)$.

- Calculate the average, $\bar{t}^p$, and the standard deviation, $\sigma_{t^p}$, of this data.

- Divide $\bar{t}^p$ and $\sigma_{t^p}$ by $p$ to get average time of a single period, $\bar{T}$ and standard deviation of a single period $\sigma_T$.

- Calculate SDOM, $\sigma_T = \frac{\sigma_T}{\sqrt{N}}$.

- Now you should have $\bar{T} \pm \sigma_T$ for your data at $\theta_i$.

- Repeat these calculations for data at each release angle.

Grading rubric uploaded on website
Error Propagation - example

We saw earlier how to determine the acceleration of gravity, $g$.

Using a simple pendulum, measuring its length and period:
- Length $l$ : $l = l_{best} \pm \delta l$
- Period $T$ : $T = T_{best} \pm \delta T$

Determine $g$ by solving: $g = l \cdot (2\pi / T)^2$

The question is what is the resulting uncertainty on $g$, $\delta g$ ??
Example

Given that:  
\[ l = 10 \pm 0.1 \text{ m} \]
\[ \alpha = 20 \pm 3^\circ \]

\[ l = 9.4 \pm 0.2 \text{ m} \]

\[ h = l \cdot \cos \alpha = 10 \cdot \cos 20^\circ = 10 \cdot 0.94 = 9.4 \text{ m} \]

\[ \delta h = \sqrt{\left( \frac{\partial h}{\partial l} \delta l \right)^2 + \left( \frac{\partial h}{\partial \alpha} \delta \alpha \right)^2} \]

\[ \frac{\partial h}{\partial l} = \cos \alpha \]

\[ \frac{\partial h}{\partial \alpha} = l \cdot (-\sin \alpha) \]

\[ \delta \alpha = 3^\circ = \frac{2\pi \text{ rad}}{360^\circ} \cdot 3^\circ = 0.05 \text{ rad} \]

\[ \delta h = \sqrt{(\cos \alpha \cdot \delta l)^2 + (l \cdot (-\sin \alpha) \cdot \delta \alpha)^2} = \sqrt{(0.94 \cdot 0.1)^2 + (10 \cdot [-0.34] \cdot 0.05)^2} = 0.2 \text{ m} \]

always use radians when calculating the errors on trig functions

From Yagil
Propagating Errors for Experiment 1

\[ \rho = \frac{3}{4\pi} \frac{g}{GR_e} \quad \text{Formula for density.} \]

\[ \sigma_\rho = \frac{3}{4\pi} \frac{1}{GR_e} \sigma_g \oplus \frac{-3}{4\pi} \frac{g}{GR_e^2} \sigma_{R_e} \quad \text{Take partial derivatives and add errors in quadrature} \]

Or, in terms of relative uncertainties: 

\[ \frac{\sigma_\rho}{\rho} = \frac{\sigma_g}{g} \oplus \frac{\sigma_{R_e}}{R_e} \]

Shorthand notation for quadratic sum: 

\[ \sqrt{a^2 + b^2} = a \oplus b \]
Propagating Errors for $R_e$

\[ R_e = \frac{2C}{\omega^2} \left( \sqrt{h_1} - \sqrt{h_2} \right)^2 \]

**basic formula**

\[ \sigma_{R_e} = \frac{\partial R_e}{\partial \Delta t} \sigma_{\Delta t} \oplus \frac{\partial R_e}{\partial h_1} \sigma_{h_1} \oplus \frac{\partial R_e}{\partial h_2} \sigma_{h_2} \]

**Propagate errors (use shorthand for addition in quadrature)**

\[ \sigma_{R_e} = \frac{2R_e}{\Delta t} \sigma_{\Delta t} \oplus \frac{R_e}{\sqrt{h_1 \left( \sqrt{h_1} - \sqrt{h_2} \right)}} \sigma_{h_1} \oplus \frac{R_e}{\sqrt{h_2 \left( \sqrt{h_1} - \sqrt{h_2} \right)}} \sigma_{h_2} \]

Note that the error blows up at $h_1=0$ and at $h_2=0$.

From Yagil
Reminder

- Prepare for lab
- Read Taylor chapter 4
- Homework due next week (Jan. 18-20) - Taylor 4.6, 4.14, 4.18, 4.26 (separate sheet)
- No lecture next Monday which is Martin Luther King Day
- Next lecture (Jan. 24) on Gaussian Distributions, lab #2, confidence in data
- Homework for lab #2 starting the following week (Jan. 25-27) - read Taylor through Chapter 5 and do problems 5.2, 5.36.