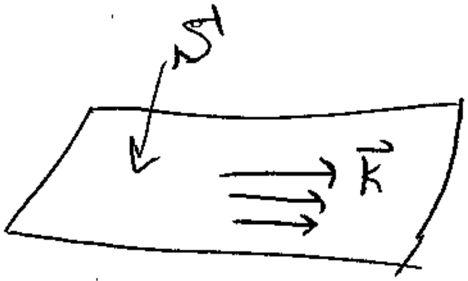
Consider surface currents $\vec{K}$



$\vec{\nabla} \cdot \vec{B} = 0$ (Gaussian pillbox)

$\Rightarrow B_\perp^{above} = B_\perp^{below}$

Now take $\oint_C \vec{B} \cdot d\vec{l}$ for C cutting through surface & normal $\parallel$ to $\vec{K}$.

$$\oint \vec{B} \cdot d\vec{l} = (B_\parallel^{above} - B_\parallel^{below}) L$$

& $I_{encl} = KL$ so $B_\parallel^{above} - B_\parallel^{below} = \mu_0 K$

e.g. take original surface S to have $\hat{n} = \hat{z}$ and $\vec{K} = K\hat{y}$, then we get

$$\vec{B}^{above} - \vec{B}^{below} = \mu_0 K \hat{x}$$

More generally

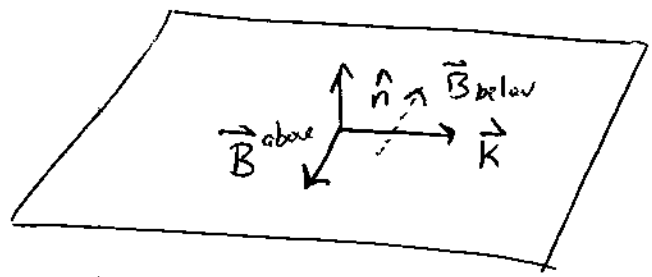
upward normal to surface

$$\boxed{\vec{B}^{above} - \vec{B}^{below} = \mu_0 \vec{K} \times \hat{n}}$$

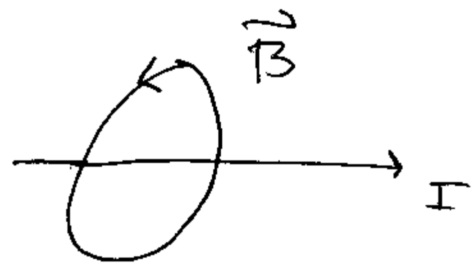
For an <u>infinite</u> plane surface S with normal $\hat{n}$ & current $\vec{K}$, we get

$$\vec{B} = \begin{cases} \frac{1}{2}\mu_0 \vec{K}\times\hat{n} & \text{above} \\ -\frac{1}{2}\mu_0 \vec{K}\times\hat{n} & \text{below} \end{cases}$$

<u>constant</u>.



Note $\vec{B}$ dir as expected by Right Hand Rule.



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} \qquad \text{"Ampere's law"}$$

always true, but not always useful (like Gauss' law). Here's a more general way to solve for $\vec{B}$:

So we assume we're given $\vec{J}$ & we want to solve $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$ for $\vec{B}$.

Can alwg do this via integral (~ superposition)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, dV'$$

Satisfies $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})$ & $\vec{\nabla}\cdot\vec{B} = 0$, as we'll soon see. For surface & line currents:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, da'$$

&

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}(\vec{r}\,') \times (\vec{r}-\vec{r}\,')\, dl'}{|\vec{r}-\vec{r}\,'|^3}$$ "Biot-Savart" law.

Solve $\vec{\nabla}\cdot\vec{B} = 0$ by $\vec{B} = \vec{\nabla}\times\vec{A}$ "Vector potential"

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}.$$

Note $\vec{A} \to \vec{A} + \vec{\nabla} f$ preserves $\vec{B} = \vec{\nabla}\times\vec{A}$ for any fn. f. "Gauge transformation"

$\vec{A} \to \vec{A} + \vec{\nabla} f$ "gauge transformation" takes on a fundamental role in realizing E & M as a local symmetry principle. Won't discuss this here.

We'll just note that we can always use this to set $\vec{\nabla} \cdot \vec{A} = 0$ "Coulomb gauge."

If we have some $\vec{A}'$ with $\vec{\nabla} \cdot \vec{A}' \neq 0$ ~~...~~ eqn. Let $\vec{A} = \vec{A}' + \vec{\nabla} f$ where $\nabla^2 f = -\vec{\nabla} \cdot \vec{A}'$

can always solve this Poisson eqn for f.

In Coulomb gauge $\vec{\nabla} \times \vec{B} = -\nabla^2 \vec{A} = \mu_0 \vec{J}$

So we want to solve $\vec{\nabla}^2 \vec{A} = -\mu_0 \vec{J}$

subject to $\vec{\nabla} \cdot \vec{A} = 0$. Just like Poisson eqns!

Solve using our Green's function: $\frac{-1}{4\pi |\vec{r} - \vec{r}'|}$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, dV'}$$ (up to gauge transf.)

For surface & line currents:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')\,da'}{|\vec{r}-\vec{r}\,'|} \quad \text{or} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}(\vec{r}\,')\,dl'}{|\vec{r}-\vec{r}\,'|}$$

These lead to "Biot Savart":

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int dV'\,\vec{J}(\vec{r}\,')\times\left(-\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}\,'|}\right)$$

$$= \frac{\mu_0}{4\pi}\int dV'\,\frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \quad ✓$$

So now we see clearly that ↑ satisfies $\vec{\nabla}\cdot\vec{B} = 0$.
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$

For ∞ straight wire: $\vec{J}(\vec{r}\,') = I\hat{z}\,\delta(x')\,\delta(y')$

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} dz'\,\frac{\hat{z}\times(\vec{r}-z'\hat{z})}{(x^2+y^2+(z-z')^2)^{3/2}} = \frac{\mu_0 I}{4\pi}s\hat{\phi}\int_{-\infty}^{\infty}\frac{dl'}{(s^2+l'^2)^{3/2}}$$

$$= \frac{\mu_0 I\hat{\phi}}{2\pi s} \quad ✓$$

$\boxed{\text{Aside:}}$ $\boxed{\text{This page is to ans. a question in lecture. } \underline{\text{Optional.}}}$

For particle of charge q

$$L = \tfrac{1}{2}m\vec{v}^2 + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$$ (Gaussian units, sorry!)

$$\hookrightarrow \vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{q}{c}\vec{A}$$

$$H = \tfrac{1}{2}m\vec{v}^2 + q\phi$$ ($\vec{B}$ does no work!)

$$\hookrightarrow H = \frac{\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2}{2m} + q\phi$$

Gauge transformation: $\vec{A} \to \vec{A} + \vec{\nabla} f$

$\phi \to \phi - \frac{1}{c}\frac{\partial f}{\partial t}$

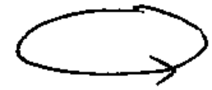
takes $L \to L + \frac{q}{c}\frac{df}{dt}$ ← $\frac{df}{dt} = \vec{v}\cdot\vec{\nabla} f + \frac{\partial f}{\partial t}$

preserves E.O.M.

In Q.M. $\psi \to \psi e^{if/\hbar}$ ← under gauge transf.

...continued...

Another example



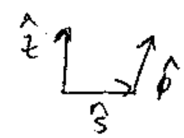
Circular current loop of radius a.

$\vec{J} = \hat{\phi} I \, \delta(z)\, \delta(s-a)$. Find $\vec{B}(\vec{r})$

for $\vec{r} = z\hat{z}$ on $\hat{z}$ axis.

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int ds'\, d\phi'\, dz'\, s' \; \frac{\hat{\phi}\, \delta(z')\, \delta(s'-a) \times (\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

$$= \frac{\mu_0 I a}{4\pi} \int d\phi' \, \frac{\hat{\phi} \times (\vec{r} - a\hat{s}')}{(a^2+z^2)^{3/2}} \qquad \leftarrow \vec{r} = a\hat{z}$$



$$= \frac{\mu_0 I a}{4\pi} \int d\phi' \, \frac{(z\hat{s} + a\hat{z})}{(a^2+z^2)^{3/2}} = \frac{\mu_0 I a^2 \hat{z}}{2(a^2+z^2)^{3/2}}$$

For $z \to 0$ center of loop $B \to \frac{\mu_0 I}{2a}\hat{z}$

for $z \to \infty$ $\vec{B} \to \frac{\mu_0 I a^2 \hat{z}}{2z^3}$ "magnetic dipole"

For current densities of any kind, we have $\vec{A}$ is continuous. But w/ surface currents:

$$\boxed{\frac{\partial \vec{A}^{above}}{\partial n} - \frac{\partial \vec{A}^{below}}{\partial n} = -\mu_0 \vec{K}}$$

Aside: For uniform $\vec{B}$, can take

$\vec{A} = -\frac{1}{2}\vec{r}\times\vec{B}$. Satisfies $\vec{\nabla}\cdot\vec{A}=0$

and $\vec{B} = \vec{\nabla}\times\vec{A}$.

If we choose our $\hat{z}$ axis so that

$\vec{B} = B\hat{z}$, we have $\vec{A} = +\frac{B}{2}\left(-y\hat{x} + x\hat{y}\right)$

e.g. Cylindrical coordinates:
$$\hat{x} = \cos\phi\,\hat{s} - \sin\phi\,\hat{\phi}$$
$$\hat{y} = \sin\phi\,\hat{s} + \cos\phi\,\hat{\phi}$$

$$\text{so } \vec{A} = \frac{B}{2}\Big(-s\sin\phi\left(\cancel{\cos\phi\,\hat{s}} - \sin\phi\,\hat{\phi}\right)$$

$$+ s\cos\phi\left(\cancel{\sin\phi\,\hat{s}} + \cos\phi\,\hat{\phi}\right)\Big) = \frac{sB}{2}\hat{\phi}$$

Check: $\vec{\nabla}\times\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)\hat{z} = B\hat{z}$ ✓

Spherical: $\vec{A} = \frac{B}{2}\Big(-r\sin\theta\sin\phi\left(\cancel{\sin\theta\cos\phi\,\hat{r}} + \right.$
$+ \cancel{\cos\theta\cos\phi\,\hat{\theta}} - \sin\phi\,\hat{\phi}\Big) + r\sin\theta\cos\phi\left(\cancel{\sin\theta\sin\phi\,\hat{r}} + \cancel{\cos\theta\sin\phi\,\hat{\theta}}\right.$
$\left. + \cos\phi\,\hat{\phi}\right)\Big) = -\frac{rB}{2}\sin\theta\,\hat{\phi}$.

Consider 2 wires

Force of I' on I is $\vec{F} = \oint_C \vec{I} \times \vec{B}' \, d\ell$

↑ $\vec{B}$ field due to C'

$$\vec{B}'(\vec{r}) = \frac{\mu_0}{4\pi} \oint_{C'} \frac{\vec{I}'(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, d\ell'$$

so $$\vec{F} = \frac{\mu_0}{4\pi} \int_C \int_{C'} \frac{\vec{I}(\vec{r}) \times \vec{I}'(\vec{r}') \times (\vec{r} - \vec{r}') \, d\ell \, d\ell'}{|\vec{r} - \vec{r}'|^3}$$