Two-level systems

2 states ψ_a, ψ_b $\quad H^0\psi_a = E_a\psi_a$
$H^0\psi_b = E_b\psi_b$

[these two states form a complete set in the Hilbert space]

$\langle \psi_a | \psi_b \rangle = \delta_{ab}$ $\quad H^0$ is time-independent

$$\psi(t=0) = c_a\psi_a + c_b\psi_b$$

With no extra perturbation:

$$\psi(t) = c_a \psi_a e^{-\frac{iE_a t}{\hbar}} + c_b \psi_b e^{-\frac{iE_b t}{\hbar}}$$

$$\psi(t) = \hat{U}(t)\,\psi(t=0)$$

$$\langle \psi | \psi \rangle = 1 \Rightarrow |c_a|^2 + |c_b|^2 = 1$$

→ We add a time-dependent perturbation $H'(t)$

$$\psi(t) = c_a(t)\,\psi_a e^{-\frac{iE_a t}{\hbar}} + c_b(t)\,\psi_b e^{-\frac{iE_b t}{\hbar}}$$

Determine $c_e(t)$ and $c_b(t)$

Solve $H\psi = i\hbar \frac{\partial \psi}{\partial t}$ $H = H^0 + H'(t)$

$$c_e \underbrace{H^0 \psi_e}_{E_e^0 \psi_e} e^{-\frac{iE_e t}{\hbar}} + c_b \underbrace{H^0 \psi_b}_{E_b^0 \psi_b} e^{-\frac{iE_b t}{\hbar}} + c_e H' \psi_e e^{-\frac{iE_e t}{\hbar}} +$$

$$+ c_b H' \psi_b e^{-\frac{iE_b t}{\hbar}} = i\hbar \Big[\dot c_e \psi_e e^{-\frac{iE_e t}{\hbar}} + \dot c_b \psi_b e^{-\frac{iE_b t}{\hbar}} +$$

$$+ c_e \psi_e \left(-\frac{iE_e}{\hbar}\right) e^{-\frac{iE_e t}{\hbar}} + c_b \psi_b \left(-\frac{iE_b}{\hbar}\right) e^{-\frac{iE_b t}{\hbar}}$$

$$\Downarrow$$

$$c_e H' \psi_e e^{-\frac{iE_e t}{\hbar}} + c_b H' \psi_b e^{-\frac{iE_b t}{\hbar}} = i\hbar \Big[\dot c_a \psi_e e^{-\frac{iE_e t}{\hbar}} + \dot c_b \psi_b e^{-\frac{iE_b t}{\hbar}}$$

Inner product with $\langle \psi_e |$

$$c_e \langle \psi_e | H' | \psi_e \rangle e^{-\frac{iE_e t}{\hbar}} + c_b \langle \psi_a | H' | \psi_b \rangle e^{-\frac{iE_b t}{\hbar}} =$$

$$= i\hbar \dot c_e e^{-\frac{iE_e t}{\hbar}} \qquad \Big/ \times \frac{-i}{\hbar} e^{+\frac{iE_e t}{\hbar}}$$

$$H'_{ij} = \langle \psi_i | H' | \psi_j \rangle \qquad \left[\left(H'_{ij}\right)^* = H'_{ji} \right]$$

$$\dot{c}_e = -\frac{i}{\hbar}\left[c_a H'_{ee} + c_b H'_{eb} e^{-i(E_b - E_a)t/\hbar} \right]$$

Similarly

$$\dot{c}_b = -\frac{i}{\hbar}\left[c_b H'_{bb} + c_e H'_{be} e^{i(E_b - E_e)t/\hbar} \right]$$

If $H'_{ee} = H'_{bb} = 0 \Rightarrow$ $(*)$ $\begin{cases} \dot{c}_e = -\frac{i}{\hbar} H'_{eb} e^{-i\omega_0 t} c_b \\ \dot{c}_b = -\frac{i}{\hbar} H'_{be} e^{i\omega_0 t} c_e \end{cases}$ coupled diff. equat.

$$\omega_0 \equiv \frac{E_b - E_e}{\hbar}$$

<u>Time-dependent perturbation theory</u>

Initial state $c_e(0) = 1$ $c_b(0) = 0$

If $H' = 0 \Rightarrow c_e^0(t) = 1$; $c_b^0(t) = 0$

<u>First-order perturbation</u>

Use $(*)$ with zeroth-order values $[H' = 0]$

$$\frac{dc_e^{(1)}}{dt} = 0 \Rightarrow c_e^{(1)}(t) = 1$$ [in general we should keep the terms H'_{ee}, H'_{bb}]

$$\left(c_e^{(1)}(t) = 1 - \frac{i}{\hbar}\int_0^t dt' H'_{ee}(t') \right)$$

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$$\frac{dC_b^{(1)}}{dt} = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} \Rightarrow C_b^{(1)} = \frac{-i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'$$

There is a finite probability $|C_b^{(1)}|^2$ that a measurement at time t the energy of the state is different than the initial value

Probability of transition from $E_a \to E_b$