Lecture 14

Solution of the time-dependent Schroedinger equation

(1) $i\hbar \dfrac{d\psi(t)}{dt} = H\psi(t)$ [let us assume to begin with that V in H does not depend on time. We'll generalize later]

with initial condition $\psi(0) = \psi_0$

linearity of (1) $\Rightarrow$ $\psi(t) = \hat{U}(t)\,\psi_0$

$\hat{U}(t)$ is a linear operator: operator of time evolution

$$\langle \hat{U}(t)\psi_0 | \hat{U}(t)\psi_0 \rangle = \langle \psi_0 | \psi_0 \rangle = 1$$ [isometric and continuous operator]

$\hat{U}(0) = 1$; $\hat{U}(t_1)\hat{U}(t_2) = \hat{U}(t_1+t_2)$ [only if $\hat{H}$ does not dep. on time]

$\psi(t) = \hat{U}(t-t_0)\,\psi_0$ [this shows that if $V(x)$ does not depend on time t_0 is not privileged]

$$\psi(x,t) = \sum_i c_i \psi_i(x,t) + \int dE\, c(E)\, \psi_E(x,t) =$$
$$= \sum_i c_i u_i(x) e^{-\frac{iE_i t}{\hbar}} + \int dE\, c(E)\, u_E(x) e^{-\frac{iEt}{\hbar}}$$

From (1)

$$i\hbar \frac{d}{dt} \hat{U}(t)\, \psi_0 = \hat{H}\, \hat{U}(t)\, \psi_0$$

Since ψ_0 is arbitrary

$$\Downarrow$$

(2) $i\hbar \frac{d}{dt} \hat{U}(t) = \hat{H}\, \hat{U}(t)$ to be solved with initial condition $\hat{U}(0) = 1$

Formal solution

$$\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}} \qquad [H \text{ ind. of time}]$$

(2) is useful when $\hat{H}$ depends on time

Rewrite (2) in integral form with $\hat{U}(0) = 1$

(3) $$\hat{U}(t) = 1 - \frac{i}{\hbar}\int_0^t dt_1\, \hat{H}(t_1)\, \hat{U}(t_1)$$

Iterate the solution (3)

Assume $\hat{U}(t_1)$ assumes the unperturbed value $\hat{U}(t_0) = 1$:

$$\hat{U}(t) \simeq 1 - \frac{i}{\hbar}\int_0^t dt_1\, \hat{H}(t_1)$$

$$\Downarrow$$

$$U(t) \simeq 1 - \frac{i}{\hbar}\int_0^t dt_1\, \hat{H}(t_1)\left[1 - \frac{i}{\hbar}\int_0^{t_1} dt_2\, \hat{H}(t_2)\right] =$$

$$= 1 - \frac{i}{\hbar}\int_0^t dt_1\, \hat{H}(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, \hat{H}(t_1)\,\hat{H}(t_2)$$

If I iterate this equation, i.e. put this solution back into (3) I obtain

$$\hat{U}(t) = \sum_{r=0}^{\infty}\left(-\frac{i}{\hbar}\right)^r \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{r-1}} dt_r\, \hat{H}(t_1)\,\hat{H}(t_2)\ldots\hat{H}(t_r)$$

Define $T\left[\hat{H}(t_1)\ \hat{H}(t_2) \ldots \hat{H}(t_r)\right]$ chronological ordered product of $\hat{H}$s

$$T\left[\hat{H}(t_1)\,\hat{H}(t_2)\right] = \begin{cases} \hat{H}(t_1)\,\hat{H}(t_2) & \text{if } t_1 \geq t_2 \\ \hat{H}(t_2)\,\hat{H}(t_1) & \text{if } t < t_2 \end{cases}$$

invariant with respect of permutation of times

$$\hat{U}(t) = \sum_{r=0}^{\infty} \frac{1}{r!}\left(-\frac{i}{\hbar}\right)^r \int_0^t dt_1 \int_0^t dt_2 \cdots \int_0^t dt_r \, T\left[\hat{H}(t_1)\,\hat{H}(t_2)\cdots\hat{H}(t_r)\right]$$

$$\hat{U}(t) = T\left\{ e^{\left[-\frac{i}{\hbar}\int_0^t dt'\,\hat{H}(t')\right]} \right\}$$

Dyson expansion. If $\hat{H}$ does not depend on t

$$\hat{U}(t) = e^{-\frac{i\hat{H}t}{\hbar}}$$

In general $\hat{U}(t, t_0) = \hat{U}(t)\,\hat{U}^{-1}(t_0) = T\left\{ e^{\left[-\frac{i}{\hbar}\int_{t_0}^t dt'\,\hat{H}(t')\right]} \right\}$

$\neq U(t-t_0)$