Lecture 5

Identical fermions

Consider an N-fermion system due to the Pauli exclusion principle the wave function of the N-particle system with single particle state $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle \ldots$ can be written as a $N\times N$ Slater determinant

$$S = \frac{1}{\sqrt{N!}} \begin{vmatrix} |\psi_1^1\rangle & |\psi_1^2\rangle & \cdots & |\psi_1^N\rangle \\ |\psi_2^1\rangle & |\psi_2^2\rangle & \cdots & |\psi_2^N\rangle \\ \vdots & & & \\ |\psi_N^1\rangle & |\psi_N^2\rangle & \cdots & |\psi_N^N\rangle \end{vmatrix}$$

So to each set $|\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_N\rangle$ of N different states chosen from among the individual states there corresponds one and only one, antisymmetrical state, represented by S

These states constitutes an orthonormal basis of a subspace $\mathcal{H}^{(A)}$ of the space $\mathcal{H}$ of all dynamical states of the whole system of N generic particles.

There are $\binom{d_n}{N_n}$ ways to choose the N_n occupied states in the single-particle energy E_n

$$Q(N_1, N_2, N_3, --) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!\,(d_n-N_n)!}$$

At thermal equilibrium and weak interactions every many-particle state with a given total energy E and N particles is equally probable.
The most probable configuration $(N_1, N_2, -)$ is the one can be achieved with largest number of ways subject to the constraints :

$$\sum_{n\geq 1}^{\infty} N_n = N$$

$$\sum_{n\geq 1}^{\infty} N_n E_n = E$$

Solve the maximization procedure:

$$N_n = \frac{d_n}{e^{(\alpha+\beta E_n)}+1}$$

α, β are Lagrange multipliers that play the role of chemical potential

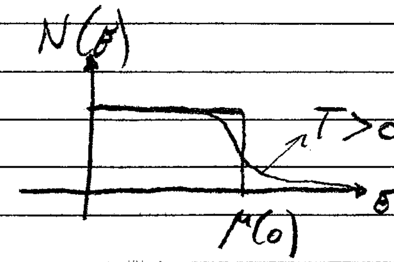
$$\mu(\overset{\text{temp}}{T}) \equiv -\alpha k_B T$$

and $\beta = \frac{1}{k_B T}$

So

$$N_n = \frac{d_n}{e^{(E_n-\mu)/k_B T}+1}$$ Fermi-Dirac distribution

For $d_n = 1$ $E_n \to E$ $N(E) \longrightarrow$



Identical bosons

Assume a gas of weakly-interacting bosons.

Symmetrization of wavefunctions require one N-particle state where a specific set of 1-particle states are occupied.

No restriction on the number of particles with same 1-particle state.

Solution
$$Q(N_1, N_2, \ldots) = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!}$$

Performing the constrained maximization:

$$N_n = \frac{d_n}{e^{(E_n - \mu)/k_B T} - 1} \qquad \text{Bose-Einstein}$$

For classical particles

$$N_n = \frac{d_n}{e^{(E_n - \mu)/k_B T}} \qquad \text{Maxwell-Boltzmann}$$