Physics 100a, Midterm

Show **all** work for full or partial credit. **Always** justify your answer. Each question or part of a question is worth 5 points, for a total of 50.

$$dv = -u\frac{dm}{m}$$

$$d\vec{l} = \hat{r}dr + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}.$$

$$\nabla F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi} = \frac{\partial F}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial F}{\partial \phi}\hat{\phi} + \frac{\partial F}{\partial z}\hat{z}.$$

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} = \frac{1}{s}\frac{\partial}{\partial s}(sF_s) + \frac{1}{s}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}.$$

$$df = \nabla F \cdot d\vec{l}, \qquad \int_V (\nabla \cdot \vec{F})dV = \oint_{\partial V} \vec{F}\cdot d\vec{a}, \qquad \int_S (\nabla \times \vec{F})\cdot d\vec{a} = \oint_{\partial S} \vec{F}\cdot d\vec{l}.$$

$$\nabla \cdot (\frac{\hat{r}}{r^2}) = 4\pi\delta^3(\vec{r})$$

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i(\vec{r}-\vec{r}_i)}{4\pi\epsilon_0|\vec{r}-\vec{r}_i|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')dV'}{|\vec{r}-\vec{r}'|^3}.$$

$$\nabla \cdot E = \rho/\epsilon_0, \qquad \nabla \times \vec{E} = 0, \qquad \vec{E} = -\nabla\phi.$$

$$\phi(\vec{r}) = -\int_O^{\vec{r}} \vec{E}\cdot d\vec{l} = \sum_i \frac{q_i}{4\pi\epsilon_0|\vec{r}-\vec{r}_i|} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}')da'}{|\vec{r}-\vec{r}'|}.$$

$$W_{1\to 2} = \int_1^2 (-Q\vec{E})\cdot d\vec{l} = Q(\phi(\vec{r}_2) - \phi\ (\vec{r}_1)).$$

$$U = \tfrac{1}{2}\sum_i q_i\phi(\vec{r}_i) = \tfrac{1}{2}\int \rho\phi dV = \frac{\epsilon_0}{2}\left(\int_V \vec{E}^2 dV + \oint_{\partial V} \phi\vec{E}\cdot d\vec{a}\right) = \frac{\epsilon_0}{2}\int \vec{E}^2 dV.$$

$$(\vec{E}_{above} - \vec{E}_{below})\cdot\hat{n} = -\frac{\partial V}{\partial n}|_{above} + \frac{\partial V}{\partial n}|_{below} = \sigma/\epsilon_0$$

$$\vec{p} = \frac{\sigma}{2}(\vec{E}_{above} + \vec{E}_{below}) = \frac{\sigma^2}{2\epsilon_0}\hat{n} = \frac{\epsilon_0}{2}\vec{E}^2\hat{n}$$

1. Consider a configuration of two infinite, parallel cylindrical shells A and B. The shells are very thin. Cylinder A has radius R, and a length L segment carries charge $\lambda_0 L$. This cylinder runs along the z axis. i.e. its central axis is $\vec{r} = (x, y, z) = (0, 0, z)$. Cylinder B has central axis $\vec{r} = (10R, 0, z)$, radius $2R$, and a length L segment has charge $-3\lambda_0 L$.

   a) What is the electric flux $\oint_S \vec{E} \cdot d\vec{a}$ for S a cylindrical surface of radius $100R$, central axis $\vec{r} = (2R, 2R, z)$, and height L along the z axis?

   b) Suppose that the cylinders are insulating shells, with surface charges uniformly distributed. Evaluate the electric field $\vec{E}$ at the point $\vec{r} = (2R, 0, 5R)$. Write $\vec{E}$ as a vector (E_x, E_y, E_z) in rectangular coordinates.

   c) Find $\vec{E} = (E_x, E_y, E_z)$ at the origin, $\vec{r} = 0$ for this configuration of charged insulating cylinders. Now suppose instead that the charged, cylindrical, thin, shells are conductors rather than insulators, what would be $\vec{E}$ at $\vec{r} = 0$ in that case?

2. Consider a solid insulating sphere, which fills the region $|\vec{r}| \leq R$, and has a spherically symmetric charge density at radius r given by $\rho(r) = \rho_0(R/r)$, with ρ_0 a constant.

   a) Find the electric field $\vec{E}$ for all $\vec{r}$.

   b) Find the scalar potential $\phi(r)$ for all $\vec{r}$, taking infinity as the zero point.

   c) What is the potential energy of this system (the work required to make it)?

   d) How much ***additional*** work is required in order to coat the outer surface of this sphere with a spherically symmetric charge distribution σ of total charge Q?

3. A metal sphere of radius R, centered at the origin, carries a total charge Q, which resides entirely on the surface. A slice is made through the sphere at height $z = 0.8R$. What is the force of repulsion between the top cap and the bottom region?

4. A conducting sphere of radius R is in an external electric field $\vec{E}_0 = E_0\hat{z}$. The potential for $|\vec{r}| \geq R$ is found to be

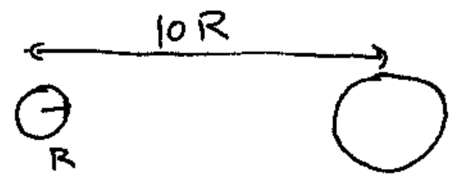
$$V(r) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$

   Find the induced surface charge $\sigma(\theta)$ on the surface of the sphere.

5. Suppose that the conducting sphere of the previous problem has an empty cavity, centered at $r = 0$, of radius $R/10$. Find the induced surface charge $\sigma_c(\theta)$ in the cavity.

Midterm Sol'ns

(1) View from above:



a) S encloses both cylinders.

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_0} = \boxed{-2\lambda_0 L}$$

b) For a single cylinderical shell, $\frac{\text{charge}}{\text{length}} = \lambda$, centered at origin, Gauss law $\Rightarrow$

$$\vec{E}_{in} = 0 \qquad \vec{E}_{out} = \frac{\lambda}{2\pi s \varepsilon_0}\,\hat{s}$$

s = distance to central axis

Point $(2R, 0, 5R)$ is in between two cylinders, use superposition:

$$\vec{E} = \vec{E}_A + \vec{E}_B, \qquad \vec{E}_A = \frac{\lambda_0\,\hat{x}}{2\pi(2R)\varepsilon_0}$$

$$\vec{E}_B = \frac{(-3\lambda_0)(-\hat{x})}{2\pi(8R)\varepsilon_0}$$

so $$\vec{E} = \frac{\lambda_0\,\hat{x}}{(2\pi R)2\varepsilon_0}\left(1 + \frac{3}{4}\right) = \boxed{\frac{7\lambda_0\,\hat{x}}{16\pi R\varepsilon_0}}$$

c) At $\vec{r} = 0$ just get electic field from cylinder B: $\vec{E} = \frac{(-3\lambda_0)(-\hat{x})}{(2\pi)(10R)\varepsilon_0} = \boxed{\frac{3\lambda_0 \hat{x}}{20\pi R \varepsilon_0}}$

(2) a) Gauss law $\Rightarrow$ $E_r 4\pi r^2 = \frac{1}{\varepsilon_0}\int_0^r \rho(r)\,dV$

$$= \frac{4\pi\rho_0}{\varepsilon_0}\int_0^{r,R} \frac{R}{r} r^2 dr = \begin{cases} \frac{4\pi\rho_0 R}{\varepsilon_0}\frac{r^2}{2} & r \le R \\ \frac{4\pi\rho_0 R^3}{2\varepsilon_0} & r \ge R \end{cases}$$

so $\vec{E}(r) = E_r(r)\hat{r}$ with

(a) $$\boxed{E_r(r) = \begin{cases} \frac{\rho_0 R}{2\varepsilon_0} & r \le R \\ \frac{\rho_0 R^3}{2\varepsilon_0 r^2} & r \ge R \end{cases}}$$

(b) $$\boxed{\phi(r) = \begin{cases} -\frac{\rho_0 R}{2\varepsilon_0} r + \frac{\rho_0 R^2}{\varepsilon_0} & (r \le R) \\ \frac{\rho_0 R^3}{2\varepsilon_0 r} & (r \ge R) \end{cases}}$$

const. needed for continuity.

c) $\mathcal{U} = \frac{\varepsilon_0}{2}\int \vec{E}^2 dV = \frac{\varepsilon_0}{2} 4\pi \int_0^\infty E_r^2 \, r^2 dr$

$$= 2\pi\varepsilon_0 \left(\int_0^R \left(\frac{\rho_0 R}{2\varepsilon_0}\right)^2 r^2 dr + \int_R^\infty \left(\frac{\rho_0 R^3}{2\varepsilon_0}\right)^2 \frac{dr}{r^2} \right)$$

$$= \frac{\rho_0^2 R^2 \pi}{2\varepsilon_0} \left(\frac{R^3}{3} + R^3 \right) = \boxed{\frac{2\pi\rho_0^2 R^5}{3\varepsilon_0}}$$

d) Suppose surface coated with charge q,

then $\phi(R) = \frac{\rho_0 R^2}{2\varepsilon_0} + \frac{q}{4\pi\varepsilon_0 R}$

Work to bring in extra charge dq is

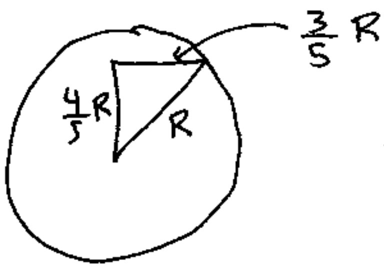
$$dW = dq\left(\phi(R) - \phi(\infty)\right) = \left(\frac{\rho_0 R^2}{2\varepsilon_0} + \frac{q}{4\pi\varepsilon_0 R}\right) dq$$

$$\int_0^R dW = \boxed{\frac{Q\rho_0 R^2}{2\varepsilon_0} + \frac{Q^2}{8\pi\varepsilon_0 R}}$$

(3) $\vec{P} = \frac{\sigma^2}{2\varepsilon_0}\hat{r} = \frac{1}{2\varepsilon_0}\left(\frac{Q}{4\pi R^2}\right)^2 \hat{r}$

Force is z component only, integrated over da

equal to pressure times z compt of area



$$F = \frac{Q^2}{32\pi^2\varepsilon_0 R^4}\,\pi\left(\frac{3}{5}R\right)^2$$

$$= \boxed{\left(\frac{9}{25}\right)\left(\frac{1}{32\pi\varepsilon_0}\right)\frac{Q^2}{R^2}}.$$

We can also get this by integrating P_z over cap:

$$\left(\frac{1}{2\varepsilon_0}\right)\frac{Q^2 R^2}{(4\pi R^2)^2}\, 2\pi \underbrace{\int_0^{\theta_0} \cos\theta \sin\theta \, d\theta}_{(x = \cos\theta)}$$

$$\int_{4/5}^{1} x\,dx = \frac{1}{2}\left(1 - \left(\frac{4}{5}\right)^2\right)$$

$$= P\pi\left(\frac{3}{5}R\right)^2$$

$$= \boxed{\frac{Q^2}{32\pi\varepsilon_0 R^2}\left(\frac{9}{25}\right)}$$ as above

(4) $$\vec{E}_\perp^{\,above} - \vec{E}_\perp^{\,below} = \frac{\sigma}{\varepsilon_0}$$

0 for conductor

$$\vec{E}_\perp^{\,above} = -\frac{\partial V^{above}}{\partial r}\bigg|_R \qquad E_\perp^{\,below} = -\frac{\partial V^{below}}{\partial r}\bigg|_R$$

So $$\sigma = \varepsilon_0 \left(E_0 (1+2)\cos\theta - 0 \right)$$

$$\boxed{\sigma = 3\,\varepsilon_0 E_0 \cos\theta}$$

(5) Since cavity is empty, $\oint \vec{E}\cdot d\vec{l} = 0$

$$\Rightarrow \quad \boxed{\sigma_c(\theta) = 0}$$