Physics 100a, Final

Show **all** work for full or partial credit. **Always** justify your answer. Each question or part of a question is worth 5 points, for a total of 100.

$$d\vec{l} = \hat{r}dr + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}.$$

$$\nabla F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi} = \frac{\partial F}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial F}{\partial \phi}\hat{\phi} + \frac{\partial F}{\partial z}\hat{z}.$$

$$\nabla \cdot \vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} = \frac{1}{s}\frac{\partial}{\partial s}(sF_s) + \frac{1}{s}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}.$$

$$df = \nabla F \cdot d\vec{l}, \qquad \int_V (\nabla \cdot \vec{F}) dV = \oint_{\partial V} \vec{F} \cdot d\vec{a}, \qquad \int_S (\nabla \times \vec{F}) \cdot d\vec{a} = \oint_{\partial S} \vec{F} \cdot d\vec{l}.$$

$$\nabla \cdot (\frac{\hat{r}}{r^2}) = 4\pi\delta^3(\vec{r})$$

$$\vec{E}(\vec{r}) = \sum_i \frac{q_i(\vec{r} - \vec{r}_i)}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|^3} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')(\vec{r} - \vec{r}')dV'}{|\vec{r} - \vec{r}'|^3}.$$

$$\nabla \cdot E = \rho/\epsilon_0, \qquad \nabla \times \vec{E} = 0, \qquad \vec{E} = -\nabla\phi.$$

$$\phi(\vec{r}) = -\int_O^{\vec{r}} \vec{E} \cdot d\vec{l} = \sum_i \frac{q_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_i|} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}')da'}{|\vec{r} - \vec{r}'|}.$$

$$W_{1\to 2} = \int_1^2 (-Q\vec{E}) \cdot d\vec{l} = Q(\phi(\vec{r}_2) - \phi(\vec{r}_1)).$$

$$U = \tfrac{1}{2}\sum_i q_i \phi(\vec{r}_i) = \tfrac{1}{2}\int \rho\phi dV = \frac{\epsilon_0}{2}\left(\int_V \vec{E}^2 dV + \oint_{\partial V} \phi\vec{E} \cdot d\vec{a}\right) = \frac{\epsilon_0}{2}\int \vec{E}^2 dV.$$

$$(\vec{E}_{above} - \vec{E}_{below}) \cdot \hat{n} = -\frac{\partial V}{\partial n}|_{above} + \frac{\partial V}{\partial n}|_{below} = \sigma/\epsilon_0$$

$$\vec{p} = \frac{\sigma}{2}(\vec{E}_{above} + \vec{E}_{below}) = \frac{\sigma^2}{2\epsilon_0}\hat{n} = \frac{\epsilon_0}{2}\vec{E}^2\hat{n}$$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{L}) \qquad C_n = \frac{2}{L}\int_0^L dx f(x)\sin(\frac{n\pi x}{L})$$

$$R_l(r) = Ar^l + \frac{B}{r^{l+1}}$$

$$f(x = \cos\theta) = \sum_{l=0}^{\infty} C_l P_l(x) \qquad C_l = \frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)dx.$$

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2 - 1), \quad P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l.$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \delta_{ll'}\frac{2}{2l+1}$$

$$4\pi\epsilon_0\phi(\vec{r}) = \frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \tfrac{1}{2}\sum_{i,j=1}^{3} Q_{ij}\frac{\hat{r}_i\hat{r}_j}{r^3} + \ldots.$$

$$Q = \int \rho(\vec{r}')dV', \quad \vec{p} = \int \vec{r}'\rho(\vec{r}')dV', \quad Q_{ij} = \int (3r'_i r'_j - r'^2\delta_{ij})\rho(\vec{r}')dV'.$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\rho(\vec{r}')dV'.$$

$$\nabla(\frac{\vec{p}\cdot\vec{r}}{r^3}) = -\frac{1}{r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right).$$

$$\mathcal{U}_d = -\vec{p}\cdot\vec{E} \qquad \vec{\tau}_d = \vec{p}\times\vec{E}$$

$$\vec{p} = \int dV\vec{P}$$

$$\sigma_b = \vec{P}\cdot\hat{n}, \qquad \rho_b = -\nabla\cdot\vec{P}.$$

$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$$

$$\nabla\cdot\vec{D} = \rho_f, \quad D_\perp^{above} - D_\perp^{below} = \sigma_f, \quad D_{||}^{above} - D_{||}^{below} = P_{||}^{above} - P_{||}^{below}.$$

$$\vec{D} = \epsilon_0(1+\chi_e)\vec{E} = \epsilon\vec{E}, \quad \epsilon = \epsilon_r\epsilon_0, \quad C = \epsilon_r C_0.$$

$$U_{macro} = \tfrac{1}{2}\int \vec{E}\cdot\vec{D}\ dV = \frac{Q^2}{2C} = \tfrac{1}{2}CV^2.$$

1. A thin rod lies along the x axis interval $0 \leq x \leq L$; the rod carries linear charge density $\lambda(x) = kx$.

   a) Evaluate exactly the potential for general height z along the $\hat{z}$ axis: $\phi(\vec{r} = z\hat{z})$.

   b) Find the total charge Q and dipole moment $\vec{p}$ of the rod.

2. A conducting sphere, of radius R, is in an external electric field $\vec{E}_{ext} = E_0\hat{z}$.

   a) Find the potential $\phi(r, \theta)$ everywhere.

   b) Find the electric field $\vec{E}$ (in spherical coordinates) just outside the conductor, at $r = R$. Draw a picture of the conductor, and this electric field just outside of it. Draw the electric field lines as arrows, including their direction. (As a check, you can compare the direction of these arrows with where you expect there to be positive or negative charges.)

   c) Suppose that a cut is made at the equator of the conductor (at $z = 0$). How much force is required to hold the two hemispheres together?

   d) Suppose now that a charge q is deposited on the conductor. Now find the new potential, everywhere.

   e) With the charge q on the conductor and the E_{ext}, find the charge density $\sigma(\theta)$ on the conductor's surface.

3. A conducting sphere of radius R has two spherical hollows cut out of it. The first hollow is of radius a_1 and it contains a charge q_1; the second hollow contains a charge q_2 and is of radius a_2. Both charges are smack dab at the center of their hollows, and their separation from each other is b.

   a) Find the surface charges at each surface of the conductor: the outer surface of radius R and the two inner spherical surfaces, of radii a_1 and a_2.

   b) What is the force of attraction between charges q_1 and q_2?

   c) How much work is required to bring a charge q_3 in from infinity, to the outer conductor's surface at radius R?

   d) Now the outer surface of the conductor (at radius R) is grounded. Write down which of the above answers to the previous parts change, and what all the new answers are for all changed quantities asked about in the previous parts.

4. A charge q_1 is at $\vec{r} = a\hat{z}$, another charge, also equal to q_1, is at $\vec{r} = a\hat{y}$, and a charge $-2q_1$ is at the origin. Approximately how much work is required to take a charge q_2 from initial position $\vec{r}_i = A\hat{z}$ to final position $\vec{r}_f = A(\hat{x} - \hat{y})$, assuming that $A >> a$? (Approximate by working to leading non-zero order in the appropriate expansion).

5. Consider two coaxial conducting cylinders, infinitely extending along the z axis, the inner one of radius $s = a$ and the outer of radius $s = c$. The space $a \leq s \leq b$ is filled with a dielectric of permittivity ϵ_1, and the space $b \leq s \leq c$ is filled with a dielectric of permittivity ϵ_2. The inner conducting cylinder (radius a) has a free charge density $\sigma_{inner} = \lambda/2\pi a$, and the outer one (radius c) has a free charge density $\sigma_{outer} = -\lambda/2\pi c$.

   a) Find the electric field $\vec{E}$ everywhere between the cylinders.

   b) Find the polarization $\vec{P}$ of both dielectrics.

   c) Find the bound charge density σ^b_{middle} at radius b, where the two dielectrics meet. (As a check, you should find that this vanishes if $\epsilon_1 = \epsilon_2$).

   d) How much energy is stored in this system per unit length? Compute the energy which is appropriate for a system with the dielectrics initially present (which includes the "spring energy" of the polarized material in the dielectrics).

6. A conducting sphere of radius a is surrounded by a spherical shell, of radius b (with $b > a$). The region between a and b is empty space, and the outer shell, of radius b, has negligible thickness. The outer shell, which is an insulator, carries some charge density $\sigma_{out}(\theta)$ (θ is the angle from the z axis, of spherical coordinates). The outer shell is measured to have voltage (relative to that of infinity) equal to $\frac{1}{2}V_0(3\cos^2\theta - 1)$.

   a) Find the potential $\phi(r, \theta)$ for all r.

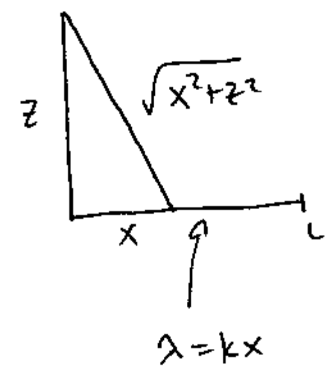
   b) Find the surface charge $\sigma_{out}(\theta)$ on the outer insulating shell.

   c) How much work is required to bring a point charge q in from $r = \infty$ to the location $\vec{r} = r\hat{x}$?

7. Consider a semi-infinite, rectangular conducting pipe, with bounding faces at $x = 0$, $x = a$, $y = 0$, $y = b$, and $z = 0$. The first four of these boundaries are at zero potential. The boundary at $z = 0$ (which isn't actually a conductor) has potential given by $\phi(x, y, 0) = V_0 \sin(\pi x/a)\sin(\pi y/b)$ for $0 \leq x \leq a$ and $0 \leq y \leq b$. Find the potential $\phi(x, y, z)$ everywhere within the pipe, i.e. all $z \geq 0$, $0 \leq x \leq a$ and $0 \leq y \leq b$.

100 a Final, Dec. 2002

(1)



$$\phi = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda\, dx}{\sqrt{x^2+z^2}}$$

$u = x^2 + z^2$

so $$\phi = \frac{k}{4\pi\varepsilon_0} \int_0^L \frac{x\,dx}{(x^2+z^2)^{1/2}} = \frac{k}{4\pi\varepsilon_0} \int_{z^2}^{z^2+L^2} \frac{du}{2} u^{-1/2}$$

(a) $\therefore$ $$\boxed{\phi(z) = \frac{k}{4\pi\varepsilon_0}\left(\sqrt{z^2+L^2} - |z|\right)}$$

(b)

$$Q = \int_0^L kx\,dx = \frac{kL^2}{2}$$

$$\vec{p} = \hat{x}\int_0^L kx^2\,dx = \frac{kL^3}{3}\,\hat{x}$$

(2) $\phi_{ext} = -E_0 z = -E_0 r\cos\theta$

Inside sphere $\phi = 0$, outside

$$\phi = \left(-E_0 r + \frac{B}{r^2}\right)\cos\theta \qquad (l = 1 \text{ only})$$

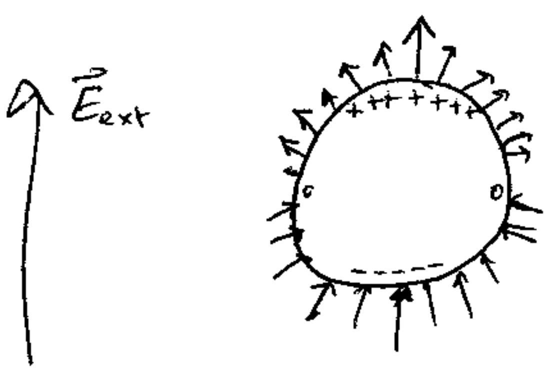
B.C. $\phi(r=R) = 0 \implies B = E_0 R^3$

So (a)
$$\boxed{\phi(r,\theta) = \begin{cases} 0 & (r \le R) \\ E_0\left(-r + \frac{R^3}{r^2}\right)\cos\theta & (r \ge R) \end{cases}}$$

(b) $\vec{E}(R,\theta) = -\nabla\left[E_0\left(-r + \frac{R^3}{r^2}\right)\cos\theta\right]\Big|_{r=R}$

$$\boxed{\vec{E}(R,\theta) = 3E_0\cos\theta\,\hat{r}}$$

$\perp$ to conductor as expected



(c) $$\frac{d\vec{F}}{dA} = \frac{\varepsilon_0}{2}\vec{E}^2\hat{r} = \frac{9}{2}\varepsilon_0 E_0^2\cos^2\theta\,\hat{r}$$

$$\vec{F} = \int_{\text{hemisphere}} \frac{d\vec{F}}{dA}\,dA = \hat{z}\,\frac{9}{2}\varepsilon_0 E_0^2 \int_{\text{hemisphere}} \cos^2\theta\underbrace{(\hat{r}\cdot\hat{z})}_{\cos\theta}\,dA$$

other compts. cancel.

$$= \hat{z}\,\frac{9}{2}\varepsilon_0 E_0^2 R^2(2\pi)\underbrace{\int_0^{\pi/2}\cos^3\theta\,\sin\theta\,d\theta}_{\int_0^1 x^3 dx = \frac{1}{4}}$$

So force needed to hold it together is

$$\boxed{\frac{9\pi}{4}\varepsilon_0 E_0^2 R^2}$$

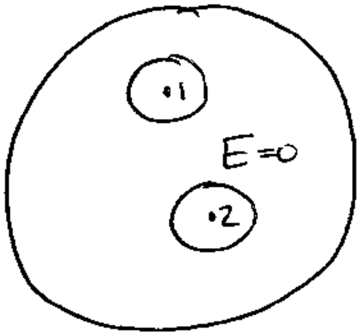
(d) Use superposition

$$\phi(r,\theta) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 R} & (r \le R) \\[2ex] \dfrac{q}{4\pi\varepsilon_0 r} + E_0\left(-r + \dfrac{R^3}{r^2}\right)\cos\theta & (r \ge R) \end{cases}$$

(e) For conductor $E_\perp^{\text{above}} = \sigma/\varepsilon_0$

$$\boxed{\sigma = \frac{q}{4\pi R^2} + 3\varepsilon_0 E_0\cos\theta}$$

(3)



(a) $\boxed{\begin{aligned} \sigma_R &= (q_1+q_2)/4\pi R^2 \\ \sigma_1 &= -q_1/4\pi a_1^2 \\ \sigma_2 &= -q_2/4\pi a_2^2 \end{aligned}}$

(b) $\boxed{\vec{F}=0}$

(c) $\boxed{W_{\infty\to R} = q_3\left(\dfrac{q_1+q_2}{4\pi\varepsilon_0 R}\right)}$

(d) Changes: now $\sigma_R = 0$ & $W_{\infty\to R} = 0$

otherwise the same

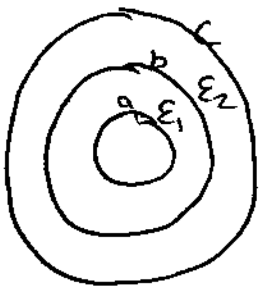
(4) $Q_{total} = 0$ $\qquad \vec{p} = q_1 a(\hat{z}+\hat{y})$

$$\phi_{dipole} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} \qquad \phi_{dipole}(\vec{r}=A\hat{z}) = \frac{q_1 a A}{4\pi\varepsilon_0 A^3}$$

$$\phi_{dipole}(\vec{r}=A(\hat{x}-\hat{y})) = \frac{-q_1 a A}{4\pi\varepsilon_0 2^{3/2} A^3}$$

So $W_{i\to f} = q_2\left(\dfrac{-q_1 a}{4\pi\varepsilon_0 2^{3/2} A^2} - \dfrac{q_1 a}{4\pi\varepsilon_0 A^2}\right)$

$$\boxed{W_{i\to f} = -\frac{q_1 q_2 a}{4\pi\varepsilon_0 A^2}\left(2^{-3/2}+1\right)}$$

(5)  Inside: $D(2\pi s L) = \lambda L$

so $D = \dfrac{\lambda \hat{s}}{2\pi s}$ inside

$D = 0$ outside

(a) $\vec{E} = 0$ for $s<a$ or $s>c$

$\vec{E} = \dfrac{\lambda \hat{s}}{2\pi s \varepsilon_1}$ for $a \le s \le b$

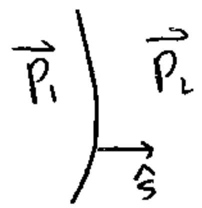
$\vec{E} = \dfrac{\lambda \hat{s}}{2\pi s \varepsilon_2}$ for $b \le s \le c$

$\vec{P} = (\varepsilon - \varepsilon_0)\vec{E} \Rightarrow$

(b) $\vec{P}_1 = \dfrac{(\varepsilon_1-\varepsilon_0)\lambda \hat{s}}{2\pi s \varepsilon_1}$ in ε_1

$\vec{P}_2 = \dfrac{(\varepsilon_2-\varepsilon_0)\lambda \hat{s}}{2\pi s \varepsilon_2}$ in ε_2

(c) $$\sigma^{b}_{middle} = (\vec{P}_1 - \vec{P}_2)\cdot\hat{s}\Big|_{s=b}$$



So $$\sigma^{b}_{middle} = \frac{\lambda}{2\pi b}\left(\left(1-\frac{\varepsilon_0}{\varepsilon_1}\right) - \left(1-\frac{\varepsilon_0}{\varepsilon_2}\right)\right)$$

$$\boxed{\sigma^{b}_{middle} = \frac{\lambda\varepsilon_0}{2\pi b}\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1\varepsilon_2}\right)}$$

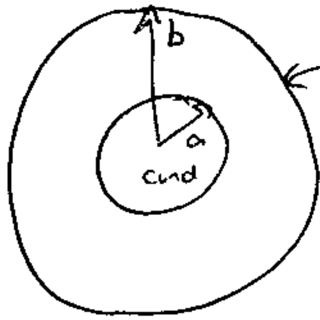
(d) $$\mathcal{U}_{macro} = \int \frac{\vec{D}^2}{2\varepsilon}\,dV = L\int \frac{\vec{D}^2}{2\varepsilon}\,s\,ds\,d\varphi$$

So $$\frac{\mathcal{U}_{macro}}{L} = 2\pi\left(\frac{\lambda}{2\pi}\right)^2\left[\int_a^b \frac{ds}{2\varepsilon_1 s} + \int_b^c \frac{ds}{2\varepsilon_2 s}\right]$$

$$\boxed{\frac{\mathcal{U}_{macro}}{L} = \frac{\lambda^2}{2\pi}\left(\frac{1}{\varepsilon_1}\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_2}\ln\left(\frac{c}{b}\right)\right)}$$

agrees w/ $\frac{\mathcal{U}}{L} = \frac{Q^2}{2CL}$ w/ $Q = \lambda L$ & $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$

(6)



$\phi = V_0 P_2(\theta)$

$$\phi = \begin{cases} 0 & r \le a \\ \left(Ar^2 + \frac{B}{r^3}\right) P_2 & a \le r \le b \\ \frac{V_0 b^3}{r^3} P_2 & r \ge b \end{cases}$$

$\ell = 2$ only

Continuity: $\quad Aa^2 + \frac{B}{a^3} = 0 \qquad Ab^2 + \frac{B}{b^3} = V_0$

$$\hookrightarrow \quad A = \frac{V_0 b^3}{b^5 - a^5} \qquad B = -\frac{V_0 a^5 b^3}{b^5 - a^5}$$

(a)

So
$$\boxed{\phi(r,\theta) = \begin{cases} 0 & r \le a \\ \frac{V_0 b^3}{(b^5 - a^5)}\left(r^2 - \frac{a^5}{r^3}\right) P_2(\cos\theta) & a \le r \le b \\ \frac{V_0 b^3}{r^3} P_2(\cos\theta) & r \ge b \end{cases}}$$

(b) $\quad E_\perp^a - E_\perp^b = \sigma / \varepsilon_0$

$$\text{so} \quad \sigma = \varepsilon_0 \underbrace{\left(-\frac{\partial}{\partial r}\right)}_{(\perp \text{ compt})} \left(\frac{V_0 b^3}{r^3} - \frac{V_0 b^3}{(b^5 - a^5)}\left(r^2 - \frac{a^5}{r^3}\right)\right) P_2 \Bigg|_{r=b}$$

So $\sigma = \varepsilon_0 V_0 \left(\frac{3}{b} + \frac{2b^4}{b^5 - a^5} + \frac{3a^5}{b(b^5-a^5)} \right) P_2(\cos\theta)$

(b)

$$\boxed{\sigma = \frac{5 V_0 \varepsilon_0 b^4}{b^5 - a^5} P_2(\cos\theta)}$$

(c) $W_{\infty \to f} = q\left(\phi(r\hat{x}) - \cancel{\phi(\infty)}^{\,0}\right) = q \frac{V_0 b^3}{r^3} P_2(\cos\theta = 0)$ $\uparrow \hat{x}$

$$\boxed{W = -\frac{q V_0 b^3}{2r^3}}$$

(7)

$$\boxed{\phi = V_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) e^{-Kz}}$$

Using separation of variables & $z \to \infty$ limit.

Plug into $\nabla^2 \phi = 0 \implies$ $\boxed{K = \sqrt{\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}}}$

Usual Fourier sum reduces to just the single term above, because the x & y dependence given, $\sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$, is already just 1 term, $n = m = 1$, of usual Fourier expansion

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right).$$