

**TURBULENT DIFFUSION OF MAGNETIC FIELDS IN WEAKLY
IONIZED GAS**

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ABSTRACT

The diffusion of uni-directional magnetic fields by two dimensional turbulent flows in a weakly ionized gas is studied. The fields here are orthogonal to the plane of fluid motion. This simple model arises in the context of the decay of the mean magnetic flux to mass ratio in the interstellar medium. When ions are strongly coupled to neutrals, the transport of a large-scale magnetic field is driven by both turbulent mixing and nonlinear, ambipolar drift. Using a standard homogeneous and Gaussian statistical model for turbulence, we show rigorously that a large-scale magnetic field can decay on at most turbulent mixing time scales when the field and neutral flow are strongly coupled. There is no enhancement of the decay rate by ambipolar diffusion. These results extend the Zeldovich theorem to encompass the regime of two dimensional flows and orthogonal magnetic fields, recently considered by Zweibel (2002). The limitation of the strong coupling approximation and its implications are discussed.

Subject headings: ISM: magnetic fields — magnetic fields — turbulence — diffusion

1. INTRODUCTION

In the interstellar medium, where the bulk of fluid consists of neutral gas, magnetic fields appear to be constantly lost from the fluid—a phenomenon often referred to as ambipolar diffusion (Spitzer 1978). It is simply because magnetic fields move with (or are tied to) ionized gases while there is slippage between the motion of ionized and neutral gases. For parameter values typical of interstellar clouds, the ambipolar drift, however, appears to be too slow (roughly by two orders of magnitude) to explain the large dispersion in the correlation between density and magnetic field strength (Zweibel 2002). This naturally motivates the exploration of other transport processes. In particular, the interaction of nonlinear, ambipolar diffusion and turbulent advective mixing is a question of obvious relevance. In this Letter, we seek to examine the interplay of these two processes and to determine a bound on the turbulent transport of magnetic fields.

While it is well known that turbulent mixing leads to a rapid diffusion of passive scalar fields, this is no longer the case for the diffusion of even a weak magnetic field (far below equipartition) in a fully ionized gas (Cattaneo & Vainshtein 1991; Gruzinov & Diamond 1994), due to the backreaction of the Lorentz force. In a weakly ionized gas, such as in the interstellar medium, the problem becomes more complicated since turbulent mixing also depends on the collision frequency between ions and neutrals as well as on the strength of magnetic fields. It is because in a weakly ionized gas, ions undergo the frictional damping due to collisions with neutrals, which effectively reduces the effect of the Lorentz force (or Alfvén waves). In fact, the previous work by Kim (1997) demonstrated that in two spatial dimensions (2D), the diffusion is still reduced below its kinematic value, but the critical strength of a large-scale magnetic field above which the diffusion is reduced can be larger (by a factor of $\sqrt{\nu_{in}\tau}$, where ν_{in} and τ are the ion-neutral collision frequency and correlation time of neutrals) as compared to what happens in the case of a fully ionized gas.

For simplicity, in this Letter, we consider the mixing of uni-directional magnetic fields by two dimensional flows which are perpendicular to these fields. Note that this configuration is different from traditional 2D MHD, where the fields and flows are coplanar. To maintain this geometry, it is necessary to make the strong coupling approximation (Spitzer 1978; Shu 1983; Zweibel 1988),

by assuming that the drift between ions and neutrals is balanced by the Lorentz force on ions due to frequent ion-neutral collisions (frictional damping). Our work is the generalization of Zweibel (2002) who considered the diffusion of magnetic fields by highly idealized flows made up of an ensemble of hyperbolic stagnation points. Since it is not altogether clear how to relate these flows to realistic turbulence models, we take a statistical approach here and rigorously derive the diffusion rate by assuming a standard scenario of Gaussian and homogeneous turbulence. Note that it is possible that this simplified statistical model may fail when the nonlinear, ambipolar diffusion is dominant, in which sharp front-like structures are generated (Brandenburg & Zweibel 1994).

Under the strong coupling approximation, magnetic fields are advected passively by neutral flows and diffused by nonlinear ambipolar drift (in addition to the usual Ohmic diffusion). Thus, in view of this nonlinear diffusion, one may naively expect that a large-scale magnetic field would decay at a rate which is significantly enhanced over the turbulent (kinematic) value. We show, however, that it is not the case, because of strong fluctuations (see §3). Specifically, in deriving a generalized Zeldovich theorem (which relates the macroscopic quantity (transport) to microscopic dissipation) for a weakly ionized gas in 2D motion but containing magnetic fields orthogonal to the plane of motion, we show that the flux transport due to the advection by neutral flows has an upper bound given by kinematic value while the nonlinear diffusion arising from ambipolar drift is insignificant. The remainder of the Letter is organized as follows. We present our model in §2 and derive a diffusion rate in §3. Section 4 contains the summary and discussion of the Letter.

2. MODEL

In a weakly ionized gas with $\rho \sim \rho_n \gg \rho_i$, ν_{in}/ν_{ni} is large, i.e., $\nu_{in}/\nu_{ni} = \rho_n/\rho_i \gg 1$. Here, ρ , ρ_n , and ρ_i are the density of the bulk of fluid, neutrals, and ions; and ν_{in} and ν_{ni} are ion-neutral and neutral-ion collision frequencies, respectively. Infrequent neutral-ion collisions permit us to

prescribe the motion of neutral gas, provided that

$$\frac{B^2}{4\pi\rho_i} \frac{\nu_{in}}{\nu_{ni}} (\simeq v_A^2) \ll v^2, \quad \tau\nu_{ni} \ll 1, \quad (1)$$

where $v \sim v_n$ and τ are the neutral velocity and correlation time, and $v_A = B/\sqrt{4\pi\rho}$ is the Alfvén speed with respect to the bulk of the fluid (see Kim 1997). Given a neutral velocity, the ion velocity follows simply from the strong coupling approximation as $\mathbf{v}_i = \mathbf{v}_n + [(\nabla \times \mathbf{B}) \times \mathbf{B}]/4\pi\rho_i\nu_{in}$. Note here that the strong coupling approximation is valid when

$$\nu_{in} > \tilde{\nu}_A, \quad (2)$$

where $\tilde{\nu}_A = kB/\sqrt{4\pi\rho_i} = kv_A\sqrt{\nu_{in}/\nu_{ni}}$ is the Alfvén frequency defined by using the density of ion (cf. equation [1]), with k being wave number. When this condition is violated, as is likely to be case on small scales due to the high frequency of Alfvén waves, the drift between ion and neutral motions is no longer balanced by Lorentz force, thereby requiring a self consistent treatment of ion dynamics (Kim 1997).

We consider the mixing of uni-directional magnetic fields (say, $\mathbf{B} = B(x, y)\hat{z}$) by incompressible neutral flows $\mathbf{v}(x, y)$ in the x - y plane perpendicular to these fields. Thus, we treat the 2D motions of 3D fields. Note that our problem is somewhat similar to that in the Goldreich-Sridhar model (Goldreich and Sridhar 1997), which also considers perpendicular mixing of anisotropic structures. The evolution equation for the strength of magnetic field in this geometry can be written in the following form (see also Zweibel 2002):

$$(\partial_t + \mathbf{v} \cdot \nabla)Q = \alpha \nabla \cdot [Q^2 \nabla Q]. \quad (3)$$

Here, $Q \equiv B/\rho$ and $\alpha \equiv \rho/4\pi\nu_{ni}$, and the (small) Ohmic diffusivity has been ignored. The second term on the left hand side of equation (2) represents the advection by neutral flow while the term on the right hand side is the nonlinear diffusion by ambipolar drift. The ratio of the effects of these two can be measured by ambipolar Reynolds number $R_{AD} = vl/\lambda_T = (v^2/v_A^2)(\tau\nu_{ni})$. Here, $vl = \eta_k$ is the kinematic diffusion rate, and $\lambda_T = v_A^2/\nu_{ni}$ is ambipolar drift due to total magnetic fields. Note that R_{AD} can be larger or smaller than unity, while still being consistent with equation (1).

The question is then what the total diffusion rate is in the presence of these two (advection and ambipolar drift) effects. Would they act together to significantly enhance the diffusion rate over the kinematic value η_k ? To answer this question, we assume Gaussian, homogeneous turbulence and evaluate the diffusion rate by using a quasi-linear or ‘second order smoothing’ closure (for instance, see Moffatt 1978) in the next section. The observant reader is no doubt puzzled by the fact that equation (3) is two dimensional. The motivation for this simplification is that turbulence with $\mathbf{k} \cdot \mathbf{B}_0 \neq 0$ will bend magnetic field lines, resulting in its conversion to Alfvén waves and radiation along \mathbf{B}_0 . Such fluctuations are intrinsically less effective at transporting Q , as much of their energy is expended on bending. However, flute-like eddys, with $\mathbf{k} \cdot \mathbf{B}_0 = 0$, are energetically favored for transport, and also remain correlated with the flux tube being transported for a longer time. Thus, the incorporation of eddys with finite wavenumber along \mathbf{B}_0 (i.e., $\mathbf{k} \cdot \mathbf{B}_0 \neq 0$) will reduce transport of magnetic fields.

3. Diffusion rate

We employ the decomposition of Q into large-scale $\langle Q \rangle$ and small-scale components Q' and assume that flows are on small-scales ($\langle \mathbf{v} \rangle = 0$). The equations for $\langle Q \rangle$ and Q' are then easily obtained as follows:

$$\partial_t \langle Q \rangle + \nabla \cdot \langle \mathbf{v} Q' \rangle = \alpha \nabla \cdot \langle Q^2 \nabla Q \rangle, \quad (4)$$

$$\partial_t Q' + \mathbf{v} \cdot \nabla \langle Q \rangle = \alpha \nabla \cdot \mathbf{F}, \quad (5)$$

where

$$\begin{aligned} \mathbf{F} \equiv & [Q'^2 - \langle Q'^2 \rangle] \nabla \langle Q \rangle + 2Q' \langle Q \rangle \nabla \langle Q \rangle + \langle Q \rangle^2 \nabla Q' \\ & + Q'^2 \nabla Q' - \langle Q'^2 \nabla Q' \rangle + 2 \langle Q \rangle [Q' \nabla Q' - \langle Q' \nabla Q' \rangle]. \end{aligned}$$

As can be seen from equation (4), the determination of the diffusion rate requires the computation of the flux $\Gamma_i = \langle v_i Q' \rangle$ and the cubic nonlinear term in Q . To compute the cubic nonlinear term, as well as other nonlinear terms that appear in the following analysis, we assume that the statistics

of fluctuations are Gaussian and that the turbulence is homogeneous. Then,

$$\langle Q^2 \nabla Q \rangle = [\langle Q \rangle^2 + \langle Q'^2 \rangle] \nabla \langle Q \rangle. \quad (6)$$

On the other hand, the flux Γ_i is evaluated by assuming stationary turbulence. We first multiply equation (5) by v_i , and then take the average to obtain,

$$\Gamma_i = -\frac{\eta_k}{1 + \alpha \tau k_{eff}^2 \langle Q^2 \rangle} \partial_i \langle Q \rangle = -\frac{\eta_k}{1 + R_{AD}^{-1}} \partial_i \langle Q \rangle. \quad (7)$$

Here, $\eta_k = \tau \langle v^2 \rangle / 2 \simeq vl$ is the kinematic diffusion rate; $k_{eff} \simeq 1/l$ is the inverse of the characteristic scale of fluctuating magnetic fields; $R_{AD} = \eta_k / \lambda_T$ where $\lambda_T = [\langle v_A \rangle^2 + \langle v'_A \rangle^2] / \nu_{ni}$; τ is the correlation time of fluctuating magnetic fields, which is assumed to be comparable to that of neutral velocity. Note that λ_T and R_{AD} , now defined in terms of averaged quantities, include both mean and fluctuating components. In deriving equation (7), we used $\langle v Q'^2 \rangle = 0$ and $\langle v_i v_j \rangle = \delta_{ij} \langle v^2 \rangle / 2$ by assuming an isotropic turbulence. Equation (7) states how much flux is transported from large to small scales, thereby leading to the decay of $\langle Q \rangle$. Interestingly, the diffusion rate by advection, $-\Gamma_i / \partial_i \langle Q \rangle$, has an upper bound given by kinematic diffusion η_k , and becomes smaller as R_{AD} decreases. This is because that ambipolar drift ‘renormalizes’ the correlation time so as to reduce the transport. Thus, it is clear that the kinematic turbulent flux is an upper bound on Γ_i .

In the case of stationary turbulence, the flux transport is balanced by dissipation on small scales as:

$$\Gamma_i \partial_i \langle Q \rangle = -\alpha [\langle Q \rangle^2 + \langle Q'^2 \rangle] \langle (\partial_i Q')^2 \rangle = -\lambda_T \langle (\partial_i Q')^2 \rangle. \quad (8)$$

This was obtained by multiplying equation (5) by Q' and then taking the average. Equation (8), together with (7), establishes the relation between small and large scale fields as

$$\frac{\eta_k}{1 + R_{AD}^{-1}} (\partial_i \langle Q \rangle)^2 = \lambda_T \langle (\partial_i Q')^2 \rangle. \quad (9)$$

Equation (9) is a generalized Zeldovich theorem for a weakly ionized, strongly coupled gas. It gives the relation between mean field and its gradient and relates the macroscopic quantity (i.e.,

flux transport) to microscopic dissipation. Note that the original Zeldovich theorem in a fully ionized gas can be recovered by replacing the ambipolar drift by Ohmic diffusion η ($\lambda_T \rightarrow \eta$) and by taking $R_{AD}^{-1} = 0$, which gives $\langle(\partial_i Q')^2\rangle/(\partial_i \langle Q\rangle)^2 = \eta_k/\eta = R_m$ (R_m is the magnetic Reynolds number). Of course, the situation discussed here is 3D, with fields orthogonal to the plane of 2D motion.

Finally, the (total) diffusion rate of $\langle Q\rangle$ follows from equations (5)–(7) and (9) as

$$\eta_T = \frac{\eta_k}{1 + R_{AD}^{-1}} \left[1 + \frac{|\nabla \langle Q\rangle|^2}{\langle(\nabla Q')^2\rangle} \right]. \quad (10)$$

η_T given in this form illustrates the two complementary effects of ambipolar drift on diffusion—the first is the reduction of η_T by renormalization of τ (the term R_{AD}^{-1}) and the second the enhancement of η_T by nonlinear diffusion (the second term in the square brackets). Due to the second effect, it is, in principle, possible that $\eta_T \gg \eta_k$. This, however, turns out to be very unlikely. To see this, we first estimate η_T in an interesting and more relevant case where $R_{AD} > 1$.

When $R_{AD} > 1$, equation (9) leads to

$$\frac{\langle(\nabla Q')^2\rangle}{|\nabla \langle Q\rangle|^2} \sim R_{AD} (> 1), \quad (11)$$

suggesting strong fluctuations. Equation (10) then becomes

$$\eta_T \sim \eta_k. \quad (12)$$

Thus, η_T approaches the kinematic value η_k . That is, the diffusion cannot exceed the kinematic rate, because of strong fluctuations. Note that in this limit, the field is transported as an effectively passive scalars!

We now look at a less interesting limit $R_{AD} < 1$ where the effect of turbulence does not play an important role. Note that in this limit, the strong coupling approximation can easily break down. This can be seen by rewriting the validity condition for the strong coupling approximation (equation [2]) as $R_{AD} > 1/(\tau\nu_{in})$. That is, the smaller R_{AD} (the stronger magnetic field), the easier it is to violate the strong coupling approximation. Thus, the results obtained in the limit

$R_{AD} \ll 1$ may not be consistent with this approximation. With this in mind, we reduce equation (9) to

$$\frac{\langle (\nabla Q')^2 \rangle}{|\nabla \langle Q \rangle|^2} \sim R_{AD}^2 (< 1), \quad (13)$$

for $R_{AD} < 1$. Equation (13) indicates that to satisfy the stationarity condition (equation [9]), $\langle Q \rangle^2 > \langle Q'^2 \rangle$. This follows because the dissipation due to ambipolar drift on small scales is too large to be balanced by flux transport. However, as noted previously, the strong coupling approximation (which leads to the nonlinear diffusion) is likely to be invalid on small scales, especially when magnetic fields are strong. What should happen on small scales is propagation of Alfvén waves, rather than nonlinear diffusion. Alternatively put, when the full dynamics of ions is taken into account, stationary turbulence may still be possible even when $\langle Q \rangle^2 > \langle Q'^2 \rangle$. Now, the diffusion rate in this case ($R_{AD} < 1$) follows from equations (10) and (13) as

$$\eta_T \sim \eta_k / R_{AD} = \lambda_T \simeq \langle v_A \rangle^2 / \nu_{ni}. \quad (14)$$

This is a somewhat expected result in the sense that for small R_{AD} , the diffusion rate is set by the ambipolar drift. Our non-trivial result is the observation that in this case, though the ambipolar drift due to fluctuating magnetic fields must be negligible in order to maintain stationarity.

4. SUMMARY AND DISCUSSION

The problem of transport of magnetic fields in the interstellar medium is studied by incorporating the effect of turbulence. Specifically, we consider the diffusion of uni-directional magnetic fields in the presence of 2D incompressible, turbulent (neutral) flows perpendicular to these magnetic fields, embedded in a weakly ionized gas. By assuming that the strong coupling approximation is valid on all scales, we compute the total diffusion rate of a large-scale magnetic field through a quasi-linear analysis. When the turbulence is homogeneous, stationary, and Gaussian, the diffusion rate η_T is found to depend on R_{AD} and the level of fluctuations (see equation [10]), with ambipolar drift playing two complementary roles (see §3). In particular, when $R_{AD} > 1$, η_T is shown to be at most of order the turbulent rate $\eta_k = vl$. In this case, the field is

effectively a passive scalar. Interestingly, this suggests that even in the strong coupling regime, it is unlikely that magnetic fields will diffuse at a rate faster than the simple kinematic value, in spite of the nonlinear diffusion operator. In the opposite case ($R_{AD} < 1$), we demonstrated $\eta_T \simeq \lambda_T$ as long as fluctuations are negligible compared to the mean field. Note, however, that this limit may not be consistent with the strong coupling approximation. Therefore, our result not only confirms the main point of Zweibel (2002) but also puts it on a simple, rigorous foundation.

The results of this Letter are applicable to any system with a neutral populations and weak magnetic field, such that the strong coupling approximation ($\mathbf{v}_i = \mathbf{v}_n + [(\nabla \times \mathbf{B}) \times \mathbf{B}]/4\pi\rho_i\nu_{in}$) is valid. However, because of the assumed incompressibility of neutral flows, a more uniform loading of magnetic fields discussed in the paper is basically due to the diffusion of magnetic fields in a constant density background. As the magnetic field diffuses while the density remains constant, system progresses toward a state of more uniformly loaded magnetic fields. In the turbulent case with $R_{AD} > 1$, this mass loading uniformization occurs by turbulent cascade of magnetic energy (by diffusive mixing) to small scale, where it is eliminated by Ohmic dissipation. This uniformization occurs in one large eddy turnover time, as one expects, since $\tau_{un} = \sum_n \tau_n = \sum_n l_n/v_n = \sum_n (l_n/\epsilon^{1/3}l_n^{1/3}) = (l_0^{2/3}/\epsilon^{1/3})(1/(1 - \alpha^{2/3})) \sim \tau_0$. Here, ϵ is the energy dissipation rate, and $l_n = \alpha l_{n-1}$ with $\alpha \sim 1/2$ was used. It is important to realize that the breakdown of flux freezing on small scales due to Ohmic diffusion is critical to the uniformization of loading. Strictly speaking, our results cannot be directly applied to star forming regions, where the compressibility of flows and gravity are crucial. Nevertheless, our results imply that the ambipolar drift in turbulent medium can make magnetically subcritical clouds supercritical and also that the turbulent mixing can uniformize the loading of magnetic field lines on a large eddy turnover time scale.

In the interstellar medium, the diffusion rate due to ambipolar drift alone is too small (by a few orders of magnitude) to explain the observations. Given that R_{AD} based on a large-scale magnetic field seems to be larger than unity, turbulent mixing perhaps provides a mechanism by which uniformity of the density and the strength of magnetic fields is achieved on the eddy turnover

time scale. However, ions and neutrals are unlikely to be strongly coupled on small scales due to the high frequencies of Alfvén waves, thereby invalidating the strong coupling approximation. Therefore, the complete answer to the problem ultimately requires the self-consistent treatment of ion dynamics. Furthermore, for a better estimate on R_{AD} , some information on the strength of fluctuating magnetic fields is needed. For instance, when fluctuations are much stronger than mean fields, R_{AD} based on fluctuations may be smaller than the unity, and thus ambipolar drift alone may lead to a fast diffusion. Of course, even in this case, the validity of the nonlinear (ambipolar) diffusion may become questionable due to the breakdown of the strong coupling approximation. In either cases, the relaxation of strong coupling approximation is expected to bring in the reduction of the diffusion as magnetic fields are no longer passively advected/distorted (Kim 1997). Of course, other effects, such as gravity, the detailed microscale mechanism for dissipating magnetic energy, and turbulence intermittency must be considered as well. A detailed study of this issue will be addressed in a future work.

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