Eddy Viscosity and Laminarization of Sheared Flow in 3D Reduced Magnetohydrodynamic Turbulence

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Abstract

The effect of Alfvén waves on turbulent momentum transport is studied in three dimensional reduced magnetohydrodynamic turbulence. A strong external uniform magnetic field $\mathbf{B}_0$ is assumed to be present orthogonal to both background shear flow and its inhomogeneity. Energy is injected into the fluid and/or magnetic field on small scales. It is shown analytically that eddy viscosity is reduced as $1/B_0^2$ for a strong B_0, due to the cancellation of the Reynolds stress by Maxwell stress. The sign of eddy viscosity is found to depend on the detailed properties of forcings. Specifically, it is positive for fluid forcing but depends on the anisotropy of the forcing in the case of magnetic forcing. Furthermore, it is indicated that a magnetic field tends to *laminarize* a mean shear flow. The possible implication of these results for the dynamics of the zonal flows in tokamaks is discussed in view of the cancellation of stresses.

I. INTRODUCTION

One of the most basic concepts of turbulence theory and modelling is that of eddy (turbulent) viscosity. Eddy viscosity, which represents the overall effect of momentum transport by small–scale fluctuations (turbulent momentum transport) on a mean sheared flow, has played a crucial role in the problem of momentum (or angular momentum) transport in laboratory and space plasmas, for instance, in tokamaks,[1] accretion disks,[2] and the solar interior.[3] It is well known that in two dimensional hydrodynamic turbulence (2D HD), eddy viscosity is negative, as a consequence of the conservation of enstrophy, with energy cascading from small to large scales (inverse cascade). In contrast, the energy cascade is direct in three dimensions (3D) since the vortex stretching breaks the conservation of enstrophy. In view of common occurrence of magnetic fields in laboratory and space plasmas, mentioned above, it is, however, important to incorporate the effect of magnetic fields on eddy viscosity, namely to consider magnetohydrodynamic turbulence (MHD).[4] Note that enstrophy is no longer an ideal invariant of MHD, even in 2D, on account of the Lorentz force.

In MHD, eddy viscosity originates not only from fluid (Reynolds) stress but also from magnetic (Maxwell) stress, and there is a possibility of cancellation between the two stresses.[5,6] Characteristic of a magnetized system is the dominance of Alfvén waves in turbulence, which turn random (irreversible) eddy motion into coherent (reversible) wave-like motion with an enhanced memory time (i.e., the so-called Alfvénization process). As a consequence of Alfvénization, equipartition between fluid kinetic and magnetic energies can be realized, which can then lead to the cancellation of Reynolds (fluid) stress by Maxwell (magnetic) stress. Such cancellation was indeed found in the previous works by Kim and Dubrulle who analytically derived the eddy viscosity in a 2D MHD, where a strong large–scale magnetic field and background shear flow are in parallel.[5,6] They also demonstrated laminarization of a mean shear flow by a magnetic field. The purpose of the present work is to extend these studies to the simplest 3D system by adopting 3D Reduced MHD (RMHD) and by assuming that a strong external large–scale magnetic field $B_0\hat{z}$ lies in the direction orthogonal to both large–scale shear flow $U(y)\hat{x}$ and its inhomogeneity ($\hat{y}$).

The result of this paper is directly related to the MHD drag reduction in laboratory experiments, where the introduction of external magnetic field was shown to lead to reduction in turbulent transport (eddy viscosity) as well as to laminarization of a mean shear flow.[7] Our work may also have application in the following systems. First, note that the mean field configuration that we consider in this paper is reminiscent of those in tokamaks and accretion disks. In tokamaks, a zonal flow is perpendicular to a strong equilibrium magnetic field. Here, a zonal flow is toroidally and poloidally symmetric $\mathbf{E}\times\mathbf{B}$ flow which is radially sheared and is thought to be generated by Reynolds stress in drift-wave turbulence.[1] Even if 3D RMHD does not capture all aspects of the dynamics of a zonal flow, the present work may yet shed some light on the effect of Alfvén waves on the generation of a zonal flow for high β plasmas. It may also have an implication for the (angular) momentum transport in accretion disks, which is thought to be crucial to accreting matter to a central object.[2] In addition, some relevance may be found in the drag reduction in turbulent polymer solutions where Reynolds stress can be cancelled by elastic stress (due to elastic waves), instead of Maxwell stress.[8]

In the present paper, we study momentum transport (eddy viscosity) and laminarization of a mean shear flow in 3D RMHD. We assume that the background turbulence is generated by an external forcing which injects energy into a fluid and/or magnetic field on small scales. The main aim is then to analytically calculate eddy viscosity (or turbulent viscosity), by using two-scale and quasi–linear analyses, together with the Gabor transform. We explore the dependence of eddy viscosity on the properties of forcings, by adopting either fluid or magnetic forcing under the assumption that these forcings are isotropic or anisotropic. The Gabor transform is employed here to rigorously incorporate the inhomogeneity of a background shear flow order by order, in terms of small parameter $\epsilon = l/L$, where l and L are the characteristic scales of small and large scale fields. As shall be shown later, the Gabor transform is related to the use of shearing coordinates which explicitly incorporates the shearing of eddy by a background shear in the evolution of wave number in parallel to the shear.[9] In contrast to a conventional Fourier transform, the Gabor transform allows us to study the strong shear limit where $\xi \equiv |\nu k^2/\Omega| \ll 1$, using eikonal theory. Here $\Omega = -\partial_y U(y)$ is a shear; ν is the viscosity and $k \sim 1/l$. In addition to the aforementioned shear parameter ξ, our problem has another dimensionless parameter $\gamma = |B_0 q/\Omega|$, which measures the ratio of the Alfvén frequency of mode q to its shearing rate. Here q is the wave number along $B_0\hat{z}$. Due to the complexity of the analysis, we will mainly focus on strong shear ($\xi \ll 1$) and strong magnetic field ($\gamma \gg 1$) limits, unless mentioned otherwise.

The structure of the paper is as follows. We formulate the problem in Sec. II and analytically solve the equations for fluctuations in terms of the Gabor transform in Sec. III. The result on the eddy viscosity is presented in Sec. IV. Our conclusion is provided in Sec. V. Appendices contain the summary of properties of the Gabor transform and some of the detailed algebra leading to main equations in the text.

II. GOVERNING EQUATIONS

We consider a 3D system in Cartesian coordinates where a strong uniform magnetic field $\mathbf{B}_0$ points in the z-direction ($\mathbf{B}_0 = B_0\hat{z}$); in tokamak, z represents the toroidal direction. The fluid and magnetic field are assumed to be externally stirred by small–scale random forcings. Then, the equations governing 3D RMHD are:[10]

$$\left\{\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right\}\omega = -(\mathbf{B}\cdot\nabla)\,\nabla_\perp^2 a + \nu\nabla_\perp^2\omega + F_\omega\,, \tag{1}$$

$$\left\{\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right\}a = B_0\partial_z\phi + \eta\nabla_\perp^2 a + F_a\,. \tag{2}$$

Here $\mathbf{B} = B_0\hat{z} + \mathbf{b}$ is the total magnetic field where the perturbed magnetic field $\mathbf{b}$ is related to the magnetic vector potential a as $\mathbf{b} = \nabla\times a\hat{z} = (\partial_y a, -\partial_x a, 0)$; ω is the vorticity related to the stream function ϕ as $\omega = -\nabla_\perp^2\phi$ and velocity $\mathbf{u}$ as $\omega\hat{z} = \nabla_\perp\times\mathbf{u} = (\partial_x u_y - \partial_y u_x)\hat{z}$; F_ω and F_a are small–scale random forcings acting on the fluid and magnetic field, respectively; ν and η are viscosity and Ohmic diffusion; $\nabla_\perp^2 = \partial_{xx} + \partial_{yy}$ is the two-dimensional laplacian. For the remainder of the paper, ν shall be taken to be the same as η (i.e., $\nu = \eta$) to simplify the analysis.

We assume that the velocity $\mathbf{u}$ has a large–scale component $\mathbf{U} = U(y)\hat{x}$ in the x-direction on scale L in addition to a small–scale component $\mathbf{u}'$ on scale l in the x-y plane. By the ordering of reduced MHD equations, the large–scale velocity $U(y)$ is weaker than the uniform magnetic field $B_0\hat{z}$. Since the main interest of this paper lies in the momentum transport, a magnetic field is taken to have no large–scale component in the x-y plane. Then, by assuming a scale separation between L and l ($\epsilon = l/L \ll 1$), we express the magnetic and velocity fields as follows: $\mathbf{u} = \langle \mathbf{u} \rangle + \mathbf{u}' = \mathbf{U} + \mathbf{u}'$, $\omega = \langle \omega \rangle + \omega' = \Omega + \omega'$, $\mathbf{b} = \langle \mathbf{b} \rangle + \mathbf{b}' = \mathbf{b}'$, and $a = \langle a \rangle + a' = a'$. Here the angular brackets denote an average over the statistics of the random forcings F_ω and F_a (see Sec. IV). Thus, U and $\Omega = -\partial_y U$ are large–scale fields and u', ω', b', ϕ' and a' are small–scale fields; $\langle \mathbf{u}' \rangle = \langle \omega' \rangle = \langle \mathbf{b}' \rangle = \langle a' \rangle = \langle \phi' \rangle = \langle F_\omega \rangle = \langle F_a \rangle = 0$. By neglecting local interaction terms compared to non–local terms, the equations for fluctuations can then be written in the following form:

$$\left\{ \frac{\partial}{\partial t} + U\partial_x \right\} \omega' = -B_0 \partial_z \nabla_\perp^2 a' + \nu \nabla_\perp^2 \omega' + F_\omega \,, \tag{3}$$

$$\left\{ \frac{\partial}{\partial t} + U\partial_x \right\} a' = B_0 \partial_z \phi' + \nu \nabla_\perp^2 a' + F_a \,. \tag{4}$$

For the evolution of the mean field U, we keep the nonlinear effect of small–scale fields to obtain,

$$\frac{\partial}{\partial t} U = -\partial_x \langle \pi \rangle - \partial_y \langle {u'}_x {u'}_y - {b'}_x {b'}_y \rangle + \nu \partial_{yy} U \,, \tag{5}$$

where $\pi \equiv -\left[\tilde{p} + b'^2/2\right]$ is the total pressure and $\tilde{p}$ the pressure. In the above equation, $\langle u'_x u'_y - b'_x b'_y \rangle$ is the total stress, or turbulent momentum flux, which consists of Reynolds and Maxwell stresses. It can be expressed in terms of the eddy (turbulent) viscosity ν_T as

$$\langle u'_x u'_y - b'_x b'_y \rangle = -\nu_T \partial_y U = \nu_T \Omega \,. \tag{6}$$

Note that a shear flow $U(y)\hat{x}$ shears an eddy, reducing its scale in y as time progresses. That is, the magnitude of wavenumber p increases for large t as $p(t) = p(t=0) + k\Omega t$ (see Sec. III). In a strong shear limit ($\xi \ll 1$), the shearing effect cannot be treated as a perturbation, making it inappropriate to use a conventional Fourier transform. For this reason, we adopt the Gabor transform in the following analysis, which can capture such a strong shear effect. We note that one alternative way is to use shearing coordinates.[9] The Gabor transform of a given function ψ is defined by

$$\hat{\psi}(\mathbf{k}, \mathbf{x}, t) \equiv \int d^3x' \, f(|\mathbf{x}-\mathbf{x}'|) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \psi(\mathbf{x}', t) \,, \tag{7}$$

where $f(x)$ is a filter function with a characteristic scale λ where $l \ll \lambda \ll L$ which decreases rapidly for large x. One example of such filter functions is a Gaussian filter with a form of $f(x) = \exp(-x^2/\lambda^2)$. As be seen from the above equation, the Gabor transform is a special case of the wavelet transform, and can be viewed as a localized Fourier transform with a compact support λ. (see Appendix A for a few key properties of the Gabor transform). In §3, the coupled Eqs. (3) and (4) are solved in terms of the Gabor transform. ν_T is then obtained in §4 by computing $\langle u'_x u'_y - b'_x b'_y \rangle$ in real space. Here the average is taken over the statistics of forcings.

III. FLUCTUATIONS

We denote the Gabor transforms of fluctuations $\mathbf{u}'$, ω', ϕ, $\mathbf{b}'$, a' and F by $\hat{\mathbf{u}}$, $\hat{\omega}$, $\hat{\phi}'$, $\hat{\mathbf{b}}$, $\hat{a}$ and $\hat{F}$. With the help of the properties of the Gabor transforms summarized in Appendix A, the equations for fluctuations (3) and (4) can be written in the Gabor space as follows:

$$\left[D_t + \nu(k^2+p^2)\right] \hat{\omega} = iB_0 q(k^2+p^2)\hat{a} + \hat{F}_\omega \,, \tag{8}$$

$$\left[D_t + \nu(k^2+p^2)\right] \hat{a} = \frac{iq}{k^2+p^2}\hat{\omega} B_0 + \hat{F}_a \,. \tag{9}$$

Here $\mathbf{k} = (k, p, q)$ and D_t is the total derivative defined by

$$D_t \equiv \partial_t + U\partial_x + k\Omega\partial_p \,, \tag{10}$$

where $\Omega = -\partial_y U$. Therefore, along a particle trajectory in the Gabor (eikonal) space $(\mathbf{x}, \mathbf{k}, t)$, the following relations hold: $x = x_0 + U(t-t_0)$, $y = y_0$, $z = z_0$, $k = k_0$, $p =$

$p_0 + k\Omega(t - t_0)$, and $q = q_0$. Without loss of generality, we shall assume for the remainder of the paper that $t_0 = 0$ and $\Omega = -\partial_y U > 0$.

We note that Eqs. (8) and (9) become identical to those derived in Ref. 5 if q on the right–hand sides is replaced by k. Thus, the procedure required to obtain solutions to Eqs. (8) and (9) is very similar to that described in Ref. 5. For completeness, some of intermediate steps are provided in Appendix B.

A. Solutions

According to 3D RMHD ordering, the Alfvén frequency of the mode q along a strong magnetic field B_0 is larger than the shearing rate, i.e., $\gamma \equiv |B_0 q/\Omega| \gg 1$ since $|B_0 q/\Omega| \sim (B_0/U)(L/l_z) \gg (B_0/U)(l_H/l_z) \sim O(1)$. Here l_H and l_z are characteristic radial and parallel scales of perturbation and L is the characteristic scale of a mean flow. Note that this condition $\gamma \gg 1$ can also be realized for typical tokamak parameters. Thus, the solutions for $\hat{a}(\mathbf{x}, \mathbf{k}, t)$ and $\hat{u}_y(\mathbf{x}, \mathbf{k}, t)$ are found in the case of $\gamma \gg 1$ in Appendix B, which can be written in the following form:

$$
\begin{aligned}
\hat{a}(\mathbf{x}, \mathbf{k}, t) = \alpha \int d^3x' d^3k' \int_0^t dt'\, g(\mathbf{x}, \mathbf{k}, t : \mathbf{x}', \mathbf{k}', t') \\
\times \left[\frac{i\mu\alpha'\psi'}{|\mu| k |k|} \sin\zeta \hat{F}_\omega + \left(\frac{1}{\alpha'} \cos\zeta + \frac{\alpha'\beta'\psi'}{\gamma} \sin\zeta \right) \hat{F}_a \right],
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\hat{u}_y(\mathbf{x}, \mathbf{k}, t) = -\frac{\Omega}{B_0} \int d^3x' d^3k' \int_0^t dt'\, g(\mathbf{x}, \mathbf{k}, t : \mathbf{x}', \mathbf{k}', t') \\
\times \Bigg[\frac{i\alpha'\alpha\psi'}{|\mu| k |k|} \left(-\alpha^2\beta\psi^2 \sin\zeta + \frac{\gamma}{\psi} \cos\zeta \right) \hat{F}_\omega \\
+ \frac{1}{\mu} \Bigg\{ -\alpha^3\beta\psi^2 \left(\frac{\alpha'\beta'\psi'}{\gamma} \sin\zeta + \frac{1}{\alpha'} \cos\zeta \right) \\
+ \frac{\alpha}{\psi} \left(\alpha'\beta'\psi' \cos\zeta - \frac{\gamma}{\alpha'} \sin\zeta \right) \Bigg\} \hat{F}_a \Bigg].
\end{aligned}
\tag{12}
$$

Here

$$
\mu \equiv \frac{q}{k},
$$

$$\alpha \equiv \frac{|k|}{\sqrt{k^2+p^2}}\,, \qquad \alpha' \equiv \frac{|k'|}{\sqrt{k'^2+p'^2}}\,,$$
$$\beta \equiv \frac{p}{k}\,, \qquad \beta' \equiv \frac{p'}{k'}\,,$$
$$\psi \equiv \left[1-\frac{\alpha^4}{2\gamma^2}\right]^{-1}\,, \quad \psi' \equiv \left[1-\frac{\alpha'^4}{2\gamma^2}\right]^{-1}\,,$$
$$\zeta \equiv \gamma\Omega(t-t') - \frac{1}{4\gamma}(\tan^{-1}\beta - \tan^{-1}\beta' + \alpha^2\beta - \alpha'^2\beta')\,; \tag{13}$$

$\hat{F}_\omega \equiv \hat{F}_\omega(\mathbf{x}', \mathbf{k}', t')$, $\hat{F}_a \equiv \hat{F}_a(\mathbf{x}', \mathbf{k}', t')$, and $g(x, \mathbf{k}, t : \mathbf{x}', \mathbf{k}', t')$ is a Green's function along the particle trajectory, modified by the viscosity/diffusivity and the magnetic field:

$$\begin{aligned} &g(\mathbf{x}, \mathbf{k}, t : \mathbf{x}', \mathbf{k}', t') \\ &= \delta(x - x' - U(t-t'))\delta(y-y')\delta(z-z')\delta(k-k')\delta(p-p'-k\Omega(t-t'))\delta(q-q') \\ &\times \exp\left\{\frac{\alpha^4-\alpha'^4}{4\gamma^2}\right\}\exp\left\{-\nu\left(k^2t+\frac{p^3}{3\Omega k}\right)\right\}\exp\left\{\nu\left(k'^2t'+\frac{p'^3}{3\Omega k'}\right)\right\}. \end{aligned} \tag{14}$$

We note that the solution for $\hat{u}_y$ is correct to second order in $1/\gamma$ although that for $\hat{a}$ is correct to third order. Thus, in the next section, the eddy viscosity will be calculated up to second order in $1/\gamma$.

IV. MOMENTUM FLUX AND EDDY VISCOSITY

To evaluate the total stress, we assume that the correlation function of the forcing takes the following form:[11]

$$\begin{aligned} &\langle \hat{F}_i(\mathbf{x}_1, \mathbf{k}_1, t_1)\hat{F}_j(\mathbf{x}_2, \mathbf{k}_2, t_2)\rangle \\ &\sim (2\pi)^3\delta_{ij}\delta(\mathbf{k}_1+\mathbf{k}_2)f^2(|(\mathbf{x}_1-\mathbf{x}_2)/2|)e^{i(\mathbf{x}_1-\mathbf{x}_2)\cdot\mathbf{k}_2}\tilde{\phi}_i(\mathbf{k}_2, t_2-t_1)\,, \end{aligned} \tag{15}$$

where $i, j = \omega, a$ and $\tilde{\phi}_i$ is the Fourier transform of correlation function $\phi_i(\mathbf{r}, t) \equiv \langle F_i(\mathbf{x}, t_1)F_i(\mathbf{x}+\mathbf{r}, t_2)\rangle$ that is assumed to be homogeneous. Note that the summation over index i is not implied in the above equation; $\langle F_a(\mathbf{x}, t_1)F_\omega(\mathbf{x}+\mathbf{r}, t_2)\rangle = 0$. For simplicity, we shall take $\tilde{\phi}_i(\mathbf{k}_2, t_2-t_1)$ to be delta–correlated in time, i.e,

$$\tilde{\phi}_i(\mathbf{k}_2, t_2-t_1) = \hat{\phi}_i(\mathbf{k}_2)\delta(t_2-t_1)\,. \tag{16}$$

The total stress $\langle u'_x u'_y - b'_x b'_y \rangle$ is then easily computed by using $\hat{u}_x = -p\hat{u}_y/k$ and Eqs. (12)–(16) and (A3) as:

$$
\begin{aligned}
&\langle u'_x u'_y - b'_x b'_y \rangle \\
&= -\frac{\Omega}{2B_0^2(2\pi)^3} \int d^3k \frac{k^2}{q^2} \int_\beta^\infty d\tau e^{2\overline{\chi}} \exp\left\{2\xi\left[\beta + \frac{1}{3}\beta^3\right]\right\} \exp\left\{-2\xi\left[\tau + \frac{1}{3}\tau^3\right]\right\} \\
&\times \left[\frac{\hat{\phi}_\omega(\mathbf{k})}{k^2(k^2+p^2)} \left\{\frac{2\gamma^2\tau}{1+\tau^2}\cos(2\overline{\varphi}) - \frac{2\gamma\tau^2}{(1+\tau^2)^2}\sin(2\overline{\varphi}) + \frac{\tau^3-\tau}{(1+\tau^2)^3}\right\}\right. \\
&\left.+\frac{\hat{\phi}_a(\mathbf{k})}{\alpha^2} \left\{\frac{-2\gamma^2\tau}{(1+\tau^2)}\cos(2\overline{\varphi}) + 2\gamma\sin(2\overline{\varphi})\left(\frac{\tau^2}{(1+\tau^2)^2} - \frac{2\tau\beta}{1+\tau^2}\right) + \frac{\tau^3-\tau}{(1+\tau^2)^3}\right\}\right],
\end{aligned}
\tag{17}
$$

in the limit as $t \to \infty$. Here

$$
\begin{aligned}
&\beta \equiv \frac{p}{k}, \quad \alpha \equiv \frac{1}{\sqrt{1+\beta^2}}, \quad \xi \equiv \frac{\nu k^2}{\Omega}, \\
&\overline{\varphi} \equiv \gamma(\tau - \beta) - \frac{1}{4\gamma}\left[\tan^{-1}\tau + \frac{\tau}{1+\tau^2} - \tan^{-1}\beta - \alpha^2\beta\right], \\
&\psi \equiv \left[1 - \frac{\alpha^4}{2\gamma^2}\right]^{-1}, \\
&\overline{\chi} \equiv \frac{1}{4\gamma^2}\left[\frac{1}{(1+\tau^2)^2} - \alpha^4\right].
\end{aligned}
\tag{18}
$$

For the analysis, we leave the power spectra for the forcings $\hat{\phi}_\omega$ and $\hat{\phi}_a$ unspecified and evaluate the τ-integral only in Eq. (17). It is important to remark that the dominant contribution to the τ-integral in individual Reynolds and Maxwell stresses is not shown in Eq. (17). It is because both of them have the same magnitude and therefore cancel each other in the total stress, to leading order. Note that these leading order terms are independent of B_0 and lead to a logarithmic divergence as $\xi \to 0$. The same cancellation, observed in 2D MHD,[5,6] simply reflects the fact that the fluid motion becomes dominated by Alfvén waves for a strong B_0. This cancellation is imperfect in the presence of flow shear as shown in Eq. (17). As shall be shown shortly, though, the next order non-trivial term in the total stress is inversely proportional to B_0^2.

To evaluate the τ-integral in Eq. (17), we consider the strong shear case $\xi = |\nu k^2/\Omega| \ll 1$, where the effect of shear is more important than that of viscosity/diffusivity. The opposite limit of weak shear ($\xi \gg 1$) will be briefly discussed at the end of this section. In the limit of $\xi \ll 1$, the τ-integral can easily be performed, leading to the following eddy viscosity:

$$\nu_T = \frac{1}{4B_0^2(2\pi)^3}\int d^3k \frac{k^2}{q^2}\left[\frac{k^2}{(k^2+p^2)^3}\hat{\phi}_\omega(\mathbf{k}) + \frac{2p^2-k^2}{k^2+p^2}\hat{\phi}_a(\mathbf{k})\right]. \qquad (19)$$

A few aspects of this result are of interest. First, it reveals the reduction in the momentum transport due to a magnetic field since the amplitude of eddy viscosity becomes very small as B_0 (or $\gamma = |B_0 q/\Omega|$) increases. Note that the limit $B_0 \to 0$ can not be taken in Eq. (19) since the latter is valid only when $\gamma \gg 1$. Secondly, as the amplitude of $\nu_T \propto 1/B_0^2$ is independent of a shear Ω, an equilibrium profile of a mean shear flow becomes either linear or parabolic.[5] That is, a magnetic field tends to laminarize a mean shear flow. Thirdly, Eq. (19) indicates that the sign of eddy viscosity may depend on whether the energy is injected into fluid ($\phi_\omega \neq 0$) or magnetic field ($\phi_a \neq 0$), and also whether forcings are isotropic or anisotropic. Let us now look at this interesting dependence in detail.

(i) In the limit where the energy is injected only into the fluid ($\hat{\phi}_a = 0$): ν_T is always positive irrespective of the nature of $\hat{\phi}_\omega$.

(ii) In the opposite limit with a magnetic forcing only ($\hat{\phi}_\omega = 0$): the sign of ν_T hinges on whether $\hat{\phi}_a$ is isotropic or anisotropic, as follows;

(ii-a) When $\hat{\phi}_a$ is isotropic in the x-y plane, i.e., $\hat{\phi}_a(\mathbf{k}) = \hat{\phi}_a(\sqrt{k^2+p^2}, q)$: ν_T is always negative. This can easily be shown by using cylindrical coordinates (r,θ,q) such that $d^3k = drrd\theta dq$, $k = r\cos\theta$, and $p = r\sin\theta$, and then by doing the angular (θ) part of $\mathbf{k}$–integral to yield:

$$\nu_T = -\frac{1}{32(2\pi)^2 B_0^2}\int dqdr\, \hat{\phi}_a(r,q)\, \frac{r^3}{q^2}. \qquad (20)$$

(ii-b) When $\hat{\phi}_a$ is anisotropic in x-y plane, ν_T is positive if the forcing mainly consists of components with $p/k \gg 1$ while it is negative if the opposite holds. Recalling that k and p are the wavenumbers parallel and perpendicular to the shear flow, in tokamaks, a magnetic forcing with $p/k \gg 1$ corresponds to a forcing with a poloidally elongated structure localized in r, and that with $p/k \ll 1$ represents a forcing which is radially elongated and localized in the poloidal direction. The above results are to be contrasted to the 2D MHD case,[6] where an isotropic magnetic forcing leads to a positive eddy viscosity while an anisotropic magnetic forcing can result in a negative viscosity.

Before concluding this section, we comment on the weak shear limit $\xi \gg 1$. As the effect of magnetic field is of interest to us, we assume $\gamma \gg \xi \gg 1$ so that the dissipation does not wipe out magnetic fields. In this limit, Eq. (17) can be shown to give

$$\nu_T = \frac{1}{4B_0^2(2\pi)^3}\int d^3k \frac{k^2}{q^2}\left[\frac{1}{(k^2+p^2)^2}\hat{\phi}_\omega(\mathbf{k}) - \frac{k^3(k-4p)}{(k^2+p^2)^2}\hat{\phi}_a(\mathbf{k})\right]. \tag{21}$$

Therefore, ν_T is always positive for fluid forcing, whereas it is likely to be negative for magnetic forcing.

V. CONCLUSIONS

We have studied the momentum transport in 3D RMHD. The magnetic contribution (Maxwell stress) to total stress was demonstrated to cancel out the leading order term in the Reynolds stress. This cancellation results in a total stress proportional to Ω/B_0^2 for $\gamma = |B_0 q/\Omega| \gg 1$, thereby significantly reducing the momentum transport (or eddy viscosity) for a strong magnetic field B_0. Since the amplitude of eddy viscosity $\nu_T \sim 1/B_0^2$ is independent of Ω, besides becoming very small as $\mathbf{B}_0 \to \infty$, it indicates the tendency toward effective 'laminarization' of a mean shear flow by a magnetic field. We also found that the sign of eddy viscosity depends on the properties of forcings, such as the amplitude and anisotropy. Specifically, in the strong shear limit ($\xi \ll 1$), a fluid forcing gives rise to a positive eddy viscosity, consequently energy is transferred from large to small scales. In contrast, in the case of a magnetic forcing only, the eddy viscosity is negative, unless the magnetic forcing mainly consists of eddies that are elongated in the direction parallel to the shear flow. In reality, these forcings are expected to originate from some underlying instability as well as incoherent nonlinear interaction.

The result mentioned above has a direct application to the problem of drag reduction in MHD.[7] As the basic physics underling the reduction of eddy viscosity in our problem arises from the cancellation of Reynolds stress by Maxwell stress via Alfvénization, a similar reduction is likely to occur in turbulent polymer solutions through the near cancellation of Reynolds stress by elastic stress. We note that in 3D MHD, the cancellation between total fluid and current helicities, again due to Alfvénization, is responsible for the suppression of the α effect.[12]

It is instructive to compare this result with the two dimensional hydrodynamic turbulence (2D HD), which is equivalent to the neglect of magnetic perturbation or Alfvén waves, and then to speculate on its possible implication for the dynamics of the zonal flows in tokamaks. As mentioned in Introduction, in 2D HD, the eddy viscosity is negative,[5] and accordingly, the energy is transferred from small to large scales (inverse cascade). That is, a small–scale fluctuating velocity acts as a source for the generation of a large–scale flow. As the effect of magnetic perturbations becomes important (or, going from 2D HD to 3D RMHD), the amplitude of eddy viscosity becomes small due to cancellation of total stress, and the sign of eddy viscosity can even become positive. If the eddy viscosity remains negative (as may be the case with a magnetic forcing), a large–scale flow can still grow, but at a much slower rate because of the smaller amplitude of the eddy viscosity. On the other hand, if the eddy viscosity becomes positive (as in the case of a fluid forcing), a large–scale flow will decay. Note again that in both cases, the generation of a large–scale flow is reduced due to Alfvén waves. Although the full dynamics of the zonal flows in tokamaks cannot be addressed in the framework of 3D RMHD, this result indicates the possibility of the cancellation of stresses for Alfvén waves. If modulation of the total stress is the mechanism for the generation of the zonal flows, our work suggests the reduction in the generation of zonal flows and even the decay of zonal flows by the reversal of energy cascade direction. It will be interesting to study this, by considering the modulational instability of a zonal flow in the presence of a gas of Alfvén waves and drift waves. This will be undertaken in a future paper.

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APPENDIX A: PROPERTIES OF THE GABOR TRANSFORM

A few key properties of the Gabor transform are as follows. First, the derivative of the Gabor transform can be shown to be

$$\partial_i \hat{\mathbf{u}}' \approx i k_i \hat{\mathbf{u}}' + O(1/(k\lambda)), \tag{A1}$$

where $\epsilon^* \equiv 1/(k\lambda) \ll 1$ is a small parameter: $\epsilon \ll \epsilon^* \ll 1$. Secondly, the Gabor transform of the product of a function varying over large scales (e.g. $\mathbf{U}$ and A) and a function varying over small scales (e.g. $\mathbf{u}'$ and $\mathbf{b}'$) can be expressed to first order in $\epsilon \equiv l/L$:

$$\begin{aligned} \widehat{U_j \mathbf{u}'}(\mathbf{k}, \mathbf{x}, t) &= \int f(\mathbf{x}-\mathbf{x}') e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} U_j(\mathbf{x}', t) \mathbf{u}'(\mathbf{x}', t) d\mathbf{x}'. \\ &\approx U_j(\mathbf{x}, t)\hat{\mathbf{u}}' + i\nabla_l \left(U_j(\mathbf{x}, t)\right) \nabla_{k_l} \hat{\mathbf{u}}' + O(\epsilon^2)\,, \end{aligned} \tag{A2}$$

where we used a Taylor expansion of $\mathbf{U}$ around the point $\mathbf{x}$ to first order in ϵ and an integration by parts. The Taylor expansion converges rapidly since the kernel f varies over scales of the order λ, while U varies over scales of the order L. Thirdly, the inverse transform of the Gabor transform is just an integration over all wavenumbers with a proper normalization factor:

$$\psi(\mathbf{x}, t) = \frac{1}{f(0)(2\pi)^3} \int d^3k\, \hat{\psi}(\mathbf{x}, \mathbf{k}, t)\,. \tag{A3}$$

Finally, the Gabor transform commutes with time derivative ∂_t. It also commutes with space derivative ∇ if a quantity of interest vanishes on the boundaries; otherwise, the commutation is subject to surface terms which are negligible if the region of interest is farther from the boundaries than the distance λ.

APPENDIX B: DERIVATION OF EQS. (11)–(12)

In this Appendix, we provide intermediate steps leading to Eqs. (11)–(12). First, to solve coupled equations (8)–(9), we introduce a variable

$$R = \frac{p}{k} = p_0/k + \Omega t$$

where $k = k_0$ is used. Note that $D_R = (Dt/DR)D_t = D_t/\Omega$ and $R > 0$ for $t > |p_0/k|/\Omega$. Then, Eqs. (8)–(9) can be rewritten in terms of R as

$$D_R\tilde{\omega} = i\frac{B_0}{\Omega}\mu k^3(1+R^2)\tilde{a} + \frac{1}{\Omega}\tilde{F}\,, \tag{B1}$$

$$D_R\tilde{a} = i\frac{B_0}{\Omega}\frac{\mu}{k}\frac{\tilde{\omega}}{1+R^2}\,. \tag{B2}$$

Here $\tilde{P}$ is related to $\hat{P}$ as

$$\tilde{P} \equiv \exp\left\{-\xi\left(R + \frac{R^3}{3}\right)\right\}\hat{P}\,, \tag{B3}$$

for $P = \omega$, a, and F; $\mu = q^2/k^2$. We note that the above equations becomes identical to those in Ref. 5 when $\mu = 1$. ¿From Eqs. (B1)–(B2), we can form a single equation for $\tilde{a}$ as follows:

$$D_R[(1+R^2)D_R\tilde{a}] + \gamma^2(1+R^2)\tilde{a} = \frac{i\mu B_0}{\Omega^2 k}\tilde{F}\,, \tag{B4}$$

where $\gamma \equiv |qB_0/\Omega|$. In terms of $\theta = \tan^{-1} R$, the above equation can be simplified somewhat to

$$[D_{\theta\theta} - Q]\tilde{a} = \frac{iB_0\mu}{\Omega^2 k}\sec^2\theta\tilde{F}\,, \tag{B5}$$

where $Q \equiv -\gamma^2\sec^4\theta$. We now provide a WKB solution to the above equation when $\gamma(1+R^2) \gg 1$. This condition is satisfied for all R when $\gamma \gg 1$. To solve the inhomogeneous equation (B5) in the WKB approximation, we first construct a Green's function from WKB solutions to the homogeneous equation that are correct to third order in $1/\gamma$:

$$\tilde{a} \sim \cos\theta\exp\left\{\pm i\gamma\left[\tan\theta - \frac{\theta + \sin 2\theta/2}{4\gamma^2}\right] + \frac{\cos^4\theta}{4\gamma^2}\right\},$$

as

$$G(\theta,\theta') = \Theta(\theta-\theta')\frac{\cos\theta\cos\theta'}{\gamma\,(1-\cos^4\theta'/2\gamma^2)}\sin\varphi\exp\left\{\frac{1}{4\gamma^2}[\cos^4\theta-\cos^4\theta']\right\}. \tag{B6}$$

Here $\varphi \equiv \gamma(\tan\theta - \tan\theta') - [\theta - \theta' + (\sin 2\theta - \sin 2\theta')/2]/4\gamma$, and $\Theta(x)$ is a step function. We have assumed $\hat{a} = D_R\hat{a} = \hat{\omega} = 0$ at $R = R_0$ (or $t = 0$). Then, the WKB solution to Eq. (B5), in terms of R, is

$$\begin{aligned}\tilde{a}(R,R_0) = \frac{1}{\gamma\Omega}\frac{1}{\sqrt{1+R^2}}\int_{R_0}^{R}\frac{dR'\tilde{\psi}(R')}{\sqrt{1+R'^2}}\sin\{\tilde{\varphi}(R;R')\}\exp\{\tilde{\chi}(R;R')\}\\ \times\left[\frac{iB_0}{k\Omega}\mu\tilde{F}_\omega(R') + D_{R'}[(1+R'^2)\tilde{F}_a(R')]\right],\end{aligned} \tag{B7}$$

where

$$\begin{aligned}\tilde{\psi}(R') &\equiv \left[1-\frac{1}{2\gamma^2(1+R'^2)^2}\right]^{-1},\\ \tilde{\varphi}(R;R') &\equiv \gamma(R-R') - \frac{1}{4\gamma}\left[\tan^{-1}R - \tan^{-1}R' + \frac{R}{1+R^2} - \frac{R'}{1+R'^2}\right],\\ \tilde{\chi}(R;R') &\equiv \frac{1}{4\gamma^2}\left[\frac{1}{(1+R^2)^2} - \frac{1}{(1+R'^2)^2}\right].\end{aligned}$$

The solution for $\hat{a}(\mathbf{x},\mathbf{k},t)$ can now be expressed in terms of k, $p = kR = k(R_0+\Omega t)$ and t to obtain Eq. (11) in the main text.

Next, by using $\tilde{u}_y = -i\tilde{\omega}/k(1+R^2)$ and Eq. (B2), the solution for $\hat{u}_y$ can be constructed in a similar way, resulting in Eq. (12) in the main text.

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