

Analytic Theory of L→H Transport Bifurcation for a Simple Model of Coupled Particle and Heat Fluxes

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Abstract

The transport bifurcation model of Hinton and Staebler [F.L. Hinton and G.M. Staebler, Phys. Fluids B5, 1281 (1993)] is solved analytically. A simple substitution is shown to reduce this two-field model for density and pressure evolution to two one-field bifurcation problems. The only restriction imposed is proximity to threshold. While independent evolution equations for density and pressure are obtained, both the particle and heat sources enter each equation. An exact criterion for local co-existence of L and H phase is derived, and several regularization schemes for the bifurcation problem are explored. The transition criterion is shown to be determined by the Maxwell equal area construction. Transitions in density and pressure need not occur at precisely the same radius. The implications of these results for the interpretation of hysteresis phenomena are discussed.

I. Introduction

One of the central problems in fusion plasma physics today is that of developing a quantitative understanding of the origin and structure of edge [1] and internal transport [2] barriers. While the scientific debate is ongoing, there is an emerging consensus that the incidence of such barriers indicates an intrinsic *bistability* [3] in the transport fluxes of the plasma. Loosely put, the two stable branches correspond to the ‘Low’ (L) and ‘High’ (H) confinement modes, respectively. While the details of the mechanism for transition remain controversial, nearly all candidates involve some variation on the theme of electric field shear reduction or suppression of turbulence [4,5]. Thus, nearly all models of the bi-stable flux involve the electric field shear E_r' as a fundamental control parameter.

Given the community paradigm described above, it is natural to think of the L→H transition and profile problems as *phase transition* phenomena, with the electric field shear E_r' as the order parameter [6]. Here, the L-mode is analogous to the disordered phase (i.e. one with weak or non-existent E_r' and strong turbulence) while the H-mode is analogous to the ordered phase (i.e. strong E_r' shear and weak or no turbulence). The local L→H bifurcation is then naturally analogous to a phase transition (or, more precisely, the advance or expansion of a region of H-phase into a region of L-phase [7]), and the problem of understanding H-mode pedestal structure is analogous to understanding phenomena such as phase separation in the co-existence region (i.e. as in spinodal decomposition [8]), the expansion of one phase into another, etc. Of course, the

most important questions are simply when the $L \rightarrow H$ transition and back-transition actually occur.

To this end, considerable insight has been gained by considering simple, bi-stable ‘S-curve’ bifurcation models [9,10,11]. In such models, the variation of the system state (here the local flux) as a function of the control parameter (here E_r' or a local gradient) is represented by a S-figure, with two stable branches, which respectively correspond to L and H mode. These are connected by an unstable branch or transition region (see Fig. 1). Of course, this approach is intrinsically somewhat unsatisfactory, for many reasons. These include:

- a.) the fact that the edge barrier ($L \rightarrow H$) transition is intrinsically a *two-field* problem, and must be described in terms of density and temperature, at minimum. This is because the ion pressure gradient driven electric field shear is dependent on both the density and ion temperature profiles, and so is sensitive to both particle and (ion) heat sources. This is, of course, consistent with the empirical observations that the $L \rightarrow H$ transition has both a power and a density threshold [12].
- b.) the fact that the S-curve is, itself, a function of the local plasma parameters. While the radial variation of structure of the S-curve can be represented by considering a folded surface, or flux-landscape [13] function $\Gamma(E_r', r)$, its time evolution cannot be described without the use of transport simulation.

- c.) the model is grossly over simplified, and omits essentially all of the details of the transition dynamics such as evolution of zonal [14] and mean poloidal and toroidal flows, interplay of fluctuation, flow and profile evolution, etc. [15]. It lumps all these phenomena into a simple bifurcation between L-mode and an H-mode in which all fluctuations are quenched, and transport is purely neoclassical.

In spite of these shortcomings, the one-field *S*-curve paradigm has definitely furthered our understanding of the L→H bifurcation phenomena, as it captures many key aspects of the problem in a simple paradigmatic example. Moreover, there is some evidence that the experimentally determined flux vs. gradient plot is consistent with an *S*-curve bifurcation model [16]. However, it is also true that further progress now requires a new, improved paradigm to be developed.

In this paper, we progress beyond the one-field *S*-curve paradigm to consider a *two-field* transport bifurcation model, which describes the evolution of density n and pressure p . This model is solved *analytically*. This model was first proposed, but studied primarily numerically, by Hinton and Staebler [10]. It involves a minimal model of E_r' suppression of turbulent thermal and particle transport. A simple substitution reduces the coupled equations for density and pressure to two one-field bifurcation problems. Other than proximity to threshold, no simplifying assumptions or restrictions are invoked, in contrast to previous investigations. It should be noted that while independent evolution equations for density and pressure are obtained, both particle and thermal source functions enter each equation. An exact criterion for phase co-existence (i.e. the

possibility of bifurcation) is derived. Two candidate schemes for regularization of the bifurcation problem, namely hyper-diffusivity and a variational approach, are proposed and discussed. *In both cases, we demonstrate that the actual transition criterion is that given by the Maxwell equal area construction* [17]. This differs from the criterion proposed in Refs. [9,10], which is based upon analogies with low order systems. Moreover, we demonstrate that transitions in density and pressure need not occur at precisely the same radius. The interpretation of ‘hysteresis phenomena’ in the context of this model is discussed as well. In particular, we show that the Maxwell construction transition rule precludes static, local hysteresis in the H-L back-transition. Thus, we argue that *hysteresis must be due to profile evolution effects*, requiring that the space-time dependence of the S-curve parametrization of the transport fluxes be taken into account.

Some further discussion of the bifurcation threshold and hysteresis issues is in order here. Since, by definition, bistable systems can reside in more than one stable state, the fundamental questions, are of course, which of them does the system actually prefer, and under what conditions do transitions between them occur? In the aforementioned spatial context, this problem translates into determination of the exact position of the L→H interface and of the thickness of the plasma domain in which the state with enhanced density and temperature gradients is established (i.e. the transport barrier or pedestal width). The thickness of the pedestal in turn, determines the improvement in the overall device performance through the enhancement of the density and temperature in the core plasma contained by the barrier, as well as the enhancement of the overall energy and particle confinement times (i.e. the H-factor) [18].

Mathematically, bistable systems are usually described by evolution equations, which, when temporal stationarity is imposed, produce a bi-valued relation between certain physical characteristics of the system, such as the density or temperature gradients and the respective fluxes. While the underlying physics of this nonlinear relation differs from system to system or regime to regime, the very existence of the *S*-curve structure lends the dynamics of these systems some element of universality, e.g. similar transition between branches can occur under similar conditions. At the same time, even the full time dependent description does not necessarily prescribe the time asymptotic state of the system. In other words it is by no means a priori clear which of the two stable branches of the *S*-curve the system will adopt if they occur simultaneously for given values of control parameters.

This problem can be illustrated by an example, as follows (Fig. 1). Let us take the particle deposition rate Γ inside of some closed region as a control parameter, and the density gradient ∇n at its boundary as an order parameter. Note $\nabla n/n$ is related to E_r' . Clearly, as long as Γ is sufficiently small, so is the gradient ∇n , and $\nabla n(\Gamma)$ is unique. Thus, only one (L-mode) solution occurs. For growing Γ this changes, namely $\nabla n(\Gamma)$ becomes nonlinear and two additional solutions for $\nabla n(\Gamma)$ may branch off at, say $\Gamma = \Gamma_1$, so that for $\Gamma_1 < \Gamma < \Gamma_2$ three solutions are now possible. For $\Gamma > \Gamma_2$ only one (H-mode solution) remains. The intermediate solution is usually unstable, but the uppermost is stable and the transition to the upper branch may occur (theoretically) at any value of $\Gamma_1 < \Gamma < \Gamma_2$. In low dimensional systems such as electric circuits, where the control and

order parameters are just simple time dependent variables such as voltage and current, the transition occurs where one of the stable solutions fails to exist, i.e., in our case at Γ_2 . Similarly, the reverse transition then occurs when Γ goes below the lowest possible flux, or H-mode (Γ_1), thus forming the well known “hysteresis loop”.

In continuous media (described by PDEs), where two states or phases can coexist in static, adjacent domains, this simple rule does not determine the L-H transition. Indeed, suppose that $\Gamma(x)$ is monotonically increasing, so that $\Gamma(x) > \Gamma_2 = \Gamma(x_2)$ for $x > x_2$ (Fig. 2). Hence, the H-mode definitely occupies at least the half-space $x > x_2$. Moreover, since Γ decreases to the left, the transition of the H \rightarrow L should occur (by the above rule) only at $x = x_1$ where $\Gamma(x_1) = \Gamma_1$, where the H-mode solution ceases to exist. Now, a similar consideration can be repeated in the reverse order. Namely, the L-mode must occupy the $x < x_1$ half-space and it should even persist up to $x = x_2$, where the L-branch disappears and the L \rightarrow H transition must occur. This clearly contradicts our initial assertion, except for the trivial case $x_1 = x_2$. Therefore, this method gives an ambiguous result for the transition coordinate ($x = x_1$ or $x = x_2$). Alternatively put, this example simply shows that the location of the transition layer in the interval $[x_1, x_2]$ is indeterminate. This dilemma, while not explicitly discussed in Ref. [10], was de facto circumvented there by an additional ad-hoc requirement that the jump in the density (or pressure) gradient across the transition must be the “least possible”. This ad hoc rule happened to select $x = x_1$ as the transition point. Not only is this requirement arbitrary, but there are well known elementary examples where the opposite principle applies! For instance, in a flow with a coordinate dependent velocity profile, shocks form at such

points so as to *maximize* the velocity jump across them and to *maximize* thus the efficiency of energy dissipation.

To summarize, the transition can be placed anywhere within the interval $x_1 < x < x_2$, as long as the underlying PDE admits it. Note that the model introduced in Ref. [10] and used below certainly does. Therefore, further *physical* arguments are needed to determine the coordinate of transition. The most systematic approach is to regularize the equations by adding physically relevant terms, such as a small hyper-diffusive term to the transport equations. This resolves the problem of the interface position between the L and H regions. Another useful approach is to find a variational principle for a given equation and look for the solution that minimizes the underlying functional, such as the free energy. These approaches to regularization are explored in this paper.

The remainder of this paper is organized as follows. In Section II, the basic transport model is presented. A simple substitution is then used to decouple the density and pressure equations. An exact criterion for a stationary state of phase co-existence is obtained. In Section III, two possible regularizations of the bifurcation problem are examined. The Maxwell construction rule for transitions is derived. This was first applied to the L \rightarrow H problem in Ref. [7]. In Section IV, we discuss hysteresis phenomena. Section V includes conclusions and discussion.

II. Bistable Particle and Energy Transport Model and its Stationary Solutions

Here, we present and simplify the transport model advanced by Hinton and Staebler. This model consists of two coupled diffusion equations for particle and energy transport, including independent particle and heat sources. The particle and heat fluxes are diffusive (with turbulent diffusivity) and contain a model transport suppression factor related to the effects of the sheared $\underline{E} \times \underline{B}$ flow. The $\underline{E} \times \underline{B}$ flow shear is then related to the density and pressure gradients by the force balance equation

$$\frac{dV_E}{dx} = \frac{-c}{eBn^2} \frac{dn}{dx} \frac{dp}{dx} \quad (1)$$

The effects of toroidal and poloidal mass flows are neglected. In this model, bistability originates from the multiplicative factor dependent on flow shear. For convenience, we assume slab geometry. Thus, the transport equations can be written as:

$$\frac{\partial n}{\partial t} - \frac{\partial}{\partial x} \left[D_0 + \frac{D_1}{(1 + \alpha' V_E'^2)} \right] \frac{\partial n}{\partial x} = S(x, t), \quad (2a)$$

$$\frac{3}{2} \frac{\partial p}{\partial t} - \frac{\partial}{\partial x} \left[\chi_0 + \frac{\chi_1}{(1 + \alpha' V_E'^2)} \right] \frac{\partial p}{\partial x} = H(x, t), \quad (2b)$$

with V_E' given by Eqn. (1). Both the residual transport coefficients (D_0 and χ_0), which are representative of neoclassical processes, and the turbulent transport coefficients (D_1

and χ_1) are taken constant, for simplicity. α parametrizes the level of $E \times B$ velocity shear necessary for turbulence suppression. Here $S(x,t)$ is the particle source term and $H(x,t)$ is the thermal source term. The time dependence of S and H represents the rate of gas puffing and heating power ramps, respectively. Since we are concerned primarily with edge barriers with (standard) edge fueling, $S(x)$ is taken to be localized near $x \simeq a$ while $H(x)$ is localized near $x = 0$. Note that it is the flow shear V'_E , via its dependence on density and pressure, which couples the two bistable transport equations. Note that a significant shortcoming of this model is the limitation of constant χ_1, D_1 .

Equations (2a, b) can be simplified by writing

$$g_1 = -dn/dx, \tag{3a}$$

$$g_2 = -dp/dx. \tag{3b}$$

Thus, for stationary n and p , the balance of transport and sources immediately gives:

$$D_0 g_1 + \frac{D_1 g_1}{1 + \alpha g_1^2 g_2^2} = \int_0^x S(x) dy \equiv \Gamma(x), \tag{4a}$$

$$\chi_0 g_2 + \frac{\chi_1 g_2}{1 + \alpha g_1^2 g_2^2} = \int_0^x H(x) dx \equiv Q(x). \tag{4b}$$

Here $\Gamma(x)$ and $Q(x)$ are the particle and thermal fluxes as a function of position. For edge barriers, $H(x)$ may be safely taken to be a delta function at the origin, so the spatial dependence of Q is simply constant in the barrier region. However, the spatial dependence of $\Gamma(x)$ is important, since $S(x)$ is sharply localized at the edge. As a consequence, the formation and structure of edge barriers is indeed sensitive to the *fueling profile* $S(x)$ (via $\Gamma(x)$), but depends only parametrically on the *spatially constant* heat flux Q . g_2 can then be eliminated in terms of g_1 by multiplying Eqn. (4a) by $\chi_1 g_2$, multiplying Eqn. (4b) by $D_1 g_1$, and then subtracting the results. This yields

$$g_2 = \frac{QD_1 g_1}{[\chi_1 \Gamma(x) - (D_0 \chi_1 - \chi_0 D_1) g_1]}. \quad (5)$$

Note Eqn. (5) allows us to *completely decouple* Eqn. (4a) from Eqn. (4b)!

As we are primarily interested in transition dynamics, we focus on a regime of ‘slow’ evolution in the proximity of the transition threshold. Thus, we write:

$$n = n(x, t, \tau) \quad (6a)$$

$$P = P(x, t, \tau) \quad (6b)$$

where t refers to ‘fast’ time dependence and $\tau = \epsilon t$ refers to ‘slow’ time dependence, near threshold. As we restrict consideration to cases of evolution near threshold, we

impose $\partial n/\partial t = 0$ and $\partial P/\partial t = 0$, so that the only time dynamics is the problem is ‘slow’. For consistency then, $S(x,t) = S(x,\tau)$ and $H(x,t) = H(x,\tau)$, so that the only time dependence in the sources is ‘slow’, as well. Note that the ‘slowness’ requirement constrains the time evolution rate of the sources S and H , as well. To lowest order in ε , then, the transport equations can be written as:

$$\varepsilon \frac{\partial n}{\partial \tau} = \frac{\partial}{\partial x} \left[D_0 + \frac{D_1}{(1 + \alpha' V_{E,0}'^2)} \right] \frac{\partial n}{\partial x}(x, \tau) + S(x, \tau), \quad (7a)$$

$$\frac{3}{2} \varepsilon \frac{\partial P}{\partial \tau} = \frac{\partial}{\partial x} \left[\chi_0 + \frac{\chi_1}{(1 + \alpha' V_{E,0}'^2)} \right] \frac{\partial}{\partial x} P(x, \tau) + H(x, \tau), \quad (7b)$$

where

$$V_{E,0}' = \frac{c}{eB [n(x, \tau)]^2} \frac{d}{dx} n(x, \tau) \frac{dp}{dx}(x, \tau). \quad (7c)$$

Note that Eqns. (7a,b) state that n and P are quasi-stationary. Consistent with this, and the ordering scheme, the RHS of Eqns. (7a,b) are necessarily of $o(\varepsilon)$. Finally, then, using Eqns. (3a) and Eqn. (5), the density evolution equation for $g(x, \tau)$ may be written as:

$$\varepsilon \frac{\partial g}{\partial \tau} = \frac{\partial^2}{\partial x^2} \left[g + \frac{\lambda g}{1 + \beta(x)g^4(1 + \theta(x)g)^{-2}} - \tilde{\Gamma}(x) \right]. \quad (8)$$

In Eqn. (8), the (slow) time dependence of $\Gamma(x, \tau)$ and $Q(x, \tau)$ is now ignored, and τ has been rescaled by D_0 . Here $\lambda = D_1/D_0$, $\beta(x) = \alpha' Q^2 D_1^2 / \chi_1^2 \Gamma(x)$, $\theta(x) = (\chi_0 D_1 - D_0 \chi_1) / \chi_1 \Gamma(x)$ and $\tilde{\Gamma} = \Gamma(x) / D_0$. It should be noted that the analysis in Ref. [10] was limited to the case of $\theta = 0$. The subscript “1” on g is now omitted. Note that *all explicit spatial dependence* in Eqn. (8) arises from the spatial structure of the fueling profile. Finally, a similar equation can be derived for pressure (g_2) in a precisely analogous way.

Several comments are in order at this point. First, the most severe (indeed, the only) restriction on this analysis is the condition that the evolution be ‘slow’, so there is at least an approximate balance between transport and fueling, with $\partial_t n \cong 0$, $\partial_t p \cong 0$ to lowest order. This quasi-stationarity requires that $Dn/L_n^2 \sim S$, so that $\partial_t n$ is negligible, by comparison. Similarly, if $1/T$ is the fueling ramp rate, quasi-stationarity also requires $1/T < D/L_n^2$. Similar conditions apply to the pressure. Thus, we see that the pre-transition transport and profiles together define just how slow the transition must be. We speculate, however, that faster time variation would alter only the transient dynamics and not the end-state of the bifurcation. Second, *it should be noted that while the density gradient evolution decouples from that of the pressure gradient, it does depend on the heat flux Q and the thermal conductivities χ_0, χ_1* . Also, in the case $\theta = 0$, the functional

dependence of the flux in brackets in Eqn. (8) is structurally similar to that in the one field model of Ref. [9], albeit with *significantly* different meaning of the coefficients.

For the system to achieve a stationary state, the RHS of Eqn. (8) must vanish. Thus the expression in brackets in Eqn. (8) must be either constant or linear in x . For convenience, we take both as vanishing. Equivalently, we do not consider any imposed density gradient at the boundary, and focus on the structure of $g(x)$ required by that of $S(x)$. To study the various states of the system, we need to determine the roots of

$$g + \frac{\lambda g}{1 + \beta(x)g^4(1 + \theta(x)g)^{-2}} = \tilde{\Gamma}(x). \quad (9)$$

Introducing a new variable $v = \beta^{1/4}g$, Eqn. (9) can be re-written as:

$$F(v) = v + \frac{\lambda v}{1 + v^4(1 + \Theta v)^{-2}} = \hat{\Gamma}, \quad (10)$$

where now $\Theta = \beta^{1/4}\theta = \alpha^{-1/4}(\chi_1 D_1/Q\Gamma(x))^{1/2}(\chi_0/\chi_1 - D_0/D_1)$ and $\hat{\Gamma} = \beta^{1/4}\Gamma(x)/D_0 = \alpha^{1/4}D_0^{-1}(Q\Gamma(x)D_1/\chi_1)^{1/2}$. Now, to understand the structure of solutions and bifurcations

in the solution $v = v(\hat{\Gamma}, \lambda, \Theta)$, it is useful to plot $F(v)$ in Fig. 3. Obviously, a local bifurcation (i.e. a bifurcation at some radius x which enters $\hat{\Gamma}(x)$) can occur if Eqn. (8) has more than one solution. First, this requires $F(v)$ to be non-monotonic. In the simple case of $\Theta = 0$, this requires $\lambda > \lambda_{crit} = 16/9$. In addition, L and H modes can co-exist only if the range of $\hat{\Gamma}(v)$ intersects the interval between the local minimum and local maximum of $F(v)$. This, in turn, happens when

$$\Pi_- \equiv \left(\frac{y_+}{a}\right)^{1/4} \frac{1+\lambda+y_+}{1+y_+} D_0 \sqrt{\frac{\chi_1}{D_1}} < \sqrt{Q\Gamma} < \left(\frac{y_-}{a}\right)^{1/4} \frac{1+\lambda+y_-}{1+y_-} D_0 \sqrt{\frac{\chi_1}{D_1}} \equiv \Pi_+, \quad (11a)$$

with

$$y_{\pm} = \frac{3\lambda}{2} - 1 \pm \frac{3}{2} \sqrt{\lambda \left(\lambda - \frac{16}{9} \right)} \quad (11b)$$

Therefore, in those regions where $\sqrt{Q\Gamma} < \Pi_-$ only L mode is possible, while for $\sqrt{Q\Gamma} > \Pi_+$, only H mode can exist. Another simple case is that of $v\Theta \gg 1$. The critical λ is larger in this case, i.e. $\lambda_{cr} = 8$.

Eqn. (11) gives the criterion for the *possibility* of a local bifurcation in the density gradient at a radius x , for given $S(x), H(x), D_0, D_1, \chi_0, \chi_1$ and initial conditions. Note that Eqn. (11) establishes upper and lower bounds on the product $Q\Gamma(x)$, related to heating and fueling resources, in terms of particle diffusives and thermal conduction in L and H modes. Note that the transport coefficients are fixed input parameters here. A criterion with a similar structure may be easily derived for pressure. Equation (11) may be looked at as a sort of phase-coexistence criterion for L and H modes. *It constitutes a necessary but not sufficient criterion for the $L \rightarrow H$ bifurcation.* Furthermore, it is merely *local* criterion, for a bifurcation at a particular point. It does *not* constitute a criterion for the emergence of a density pedestal of a typical width, which is the criterion of practical interest. Finally, note that Eqn. (11) states that, in general, a density pedestal requires *both* particle and heat sources in order to form.

III. Bifurcation Theory

As we have seen in the previous section, Eqn. (8) can be satisfied by a whole family of stationary solutions labeled by the point of $L \rightarrow H$ transition. It is important to understand that Eqn. (8) alone does not uniquely specify the point of transition, just the region of ‘phase coexistence’. The reason is that the expression in brackets in Eqn. (8) can vanish identically even though g jumps between the different asymptotic branches of the bifurcation diagram, Fig. 3. Such a jump is possible *anywhere* within the interval given by Eqn. (11). Hence, Eqn. (9) must be regularized. Here ‘regularization’ refers to

the addition of terms to the equation which resolve the ambiguity in the location of the transition. Since this is a nonlinear diffusion equation, a natural way to regularize it is to add a higher order diffusion term, such as hyperdiffusion.

With hyperdiffusive regularization, Eqn. (9) becomes

$$\frac{\partial g}{\partial t} = \frac{\partial^2}{\partial x^2} \left[g + \frac{\lambda g}{1 + \beta(x) g^4 (1 + \theta g)^{-2}} - \tilde{\Gamma}(x) - \varepsilon^2 \frac{\partial^2 g}{\partial x^2} \right] \quad (12)$$

where $\varepsilon > 0$ is a small parameter. The steady state solution of this equation represents a simple singular perturbation ($\varepsilon \rightarrow 0$) problem. Introducing an “inner” variable

$$\xi = \frac{x - x_f}{\varepsilon}$$

where x_f is the transition front coordinate, to the leading order in ε , we have

$$\frac{\partial^2 g}{\partial \xi^2} - g - \frac{\lambda g}{1 + \beta g^4 (1 + \theta g)^{-2}} + \tilde{\Gamma} = 0. \quad (13)$$

Here, the x -dependent coefficients should be regarded as parameters. The solution of this equation (which is the inner solution of the problem) must tend to the discontinuous outer solution as $\xi \rightarrow \pm\infty$, which are represented by the H and L roots of Eqn. (13), with $\partial^2 g / \partial \xi^2 \rightarrow 0$. Recall that here ‘discontinuous’ refers to the fact that the L and H profiles have different slopes. In other words,

$$\lim_{\xi \rightarrow \pm\infty} g(\xi, x) = g_{\pm}(x) \quad (14)$$

where g_{\pm} on the r.h.s. are the upper and lower branches of the discontinuous solution of the problem, for $\varepsilon = 0$. This asymptotic matching condition determines x_f . Now we consider the first integral of Eqn. (13) Multiplying Eqn. (13) by $\partial g / \partial \xi$ and integrating from $\xi = -\infty$ to $\xi = +\infty$ (i.e. integrating thru the regularized layer) yields

$$\int_{-\infty}^{+\infty} d\xi \left[\frac{d}{d\xi} \left[\frac{1}{2} \left(\frac{\partial g}{\partial \xi} \right)^2 \right] - \frac{\partial g}{\partial \xi} \frac{\partial \Phi}{\partial g} + \frac{\partial g}{\partial \xi} \tilde{\Gamma} \right] = 0, \quad (15a)$$

where

$$\partial\Phi/\partial g = g + \lambda g / [1 + \beta g^4 (1 + \sigma g)^{-2}]. \quad (15b)$$

Integrating the first term directly and changing variables to an integration over g in the latter two then yields

$$\frac{1}{2} \left(\frac{\partial g}{\partial \xi} \right)^2 \Big|_{-\infty}^{+\infty} + \int_{g_-}^{g_+} dg \left(\tilde{\Gamma} - \frac{\partial\Phi}{\partial g} \right) = 0. \quad (16a)$$

Since $g \rightarrow g_{\pm} = \text{const}$ as $\xi \rightarrow \pm\infty$, this result is thus equivalent to

$$\int_{g_-}^{g_+} dg \left(\tilde{\Gamma} - \partial\Phi/\partial g \right) = 0 \quad (16b)$$

Observe that g_{\pm} correspond to $\xi \rightarrow \pm\infty$, respectively.

Eqn. (16b) is simply the Maxwell construction equal area rule, familiar from the theory of first order phase transitions [17]. Eqn. (16b) states that the points g_- and g_+ on the bifurcation diagram should be chosen in such a way that the areas above and below the curve of $\partial\Phi/\partial g$, when cut by the line $\tilde{\Gamma} = \Gamma_f$, are equal (see Fig. 4). This condition uniquely determines the location of the transition front.

It is extremely important to emphasize two points here. First, the Maxwell construction rule gives a criterion for *stationarity of the transition front*, not the total disappearance of a phase. Indeed, it is obvious that front stationarity is the relevant criterion for a local transition. At each point, the Maxwell construction rule specifies the source $\tilde{\Gamma}(x)$ strength required for transition front stationarity at that point x . Moreover, front stationarity is a weaker criterion for a local $L \rightarrow H$ transition than disappearance of the L phase is, in that it requires a lower level of driving flux $\tilde{\Gamma}$.

We now take a different approach to the determination of the transition point between L and H regions. In contrast to the regularization considered above, it does not require any specific additional terms in Eqn. (8) and is therefore more general. We start with reformulating the evolution in Eqn. (8) in terms of the following variational problem.

$$\frac{\partial g}{\partial t} = \frac{\partial^2}{\partial x^2} \frac{\delta \Lambda}{\delta g} \tag{17}$$

where the “free energy” is given by the following functional

$$\Lambda = \int_0^a [\Phi(g) - \tilde{\Gamma}g] dx. \quad (18)$$

Recall that the steady state solution of Eqns. (17) and (18) is given by

$$\frac{\delta\Lambda}{\delta g} = \Phi'(g) - \tilde{\Gamma}(x) = 0. \quad (19)$$

As discussed earlier, the region involved in the transition phenomenon is where $\Phi'(g)$ is non-monotonic, i.e., where $\Gamma_1 < \Phi' < \Gamma_2$. Therefore, to isolate the transition phenomena from the possible influence of the boundaries we assume that $\tilde{\Gamma}(a) > \Gamma_2$, so only the H-mode solution is possible and realized as a stationary solution at least in some small vicinity of the right boundary. The same is assumed to be the case at the left boundary with the only difference that the L-mode solution is established there. Using these conditions at the boundaries, along with Eqns. (17, 18), we obtain

$$\frac{\partial\Lambda}{\partial t} = -\int_0^a \left\{ \frac{\partial}{\partial x} [\Phi'(g) - \tilde{\Gamma}] \right\}^2 dx \leq 0. \quad (20)$$

During the evolution to a steady state, the functional Λ is thus decreasing and

$$\frac{\partial \Lambda}{\partial t} = 0 \quad (21)$$

only when the steady state is reached. The density of Λ , which is $\Phi(g) - \tilde{\Gamma}g$, is shown in Fig. 5 for three representative values of $\tilde{\Gamma}$. To minimize Λ in steady state [see Eqn. (19)], g must be at one of the minima of this function. The functional Λ can also be considered as a function of Γ_f , the value of Γ (or, equivalently, $x = x_f$) where the transition occurs. If this transition occurs at a place other than where the minima of $\Phi(g) - \tilde{\Gamma}g$ are equal, this yields only a local minimum of Λ as a functional of g that can be reached under the constraint of fixed Γ_f . The global minimum (that includes the variation of Γ_f) is reached only if the transition occurs at a Γ_f that corresponds to the case shown in Fig. 6. In other words, while $\tilde{\Gamma}(x)$ is changing, the solution $g(x)$ should always follow the lowest of the two minima, which automatically requires that the transition occurs when they are of an equal depth. This immediately translates into the condition given by Eqn. (16b).

Now suppose that after the power-ramp phase and the temperature and pressure relaxation, only the local minimum of Λ is reached. Formally, this satisfies the steady state Eqn. (19) for an arbitrary Γ_f as long as it is between Γ_1 and Γ_2 . However, the source term $\tilde{\Gamma}$ is not strictly steady (due to noise, for example). Then, $\Phi'(g) - \tilde{\Gamma} \neq 0$ (since $\partial g / \partial t \neq 0$) and Λ in Eqn. (20) evolves towards its global minimum. We assume here for simplicity, that the fluctuating part of $\tilde{\Gamma}$ is concentrated at the transition point so

that neglecting the end-point terms in the derivation of Eqn. (20) is still justified. The global minimum can be reached only if Γ_f takes the value corresponding to that of the Maxwell rule $\Gamma_f = \Gamma_{Maxwell}$. It is interesting to note, that if Γ continues to fluctuate around its steady mean value, Λ seems to decrease further (according to Eqn. (20)) which is impossible since, by assumption, it has already reached its global minimum. The resolution of this paradox is that the boundary cannot be ignored at this stage since the induced fluctuation of g propagates to the boundaries and contribute to the limits of the integral.

The relation between the evolution of transitions in density and pressure is most clearly illustrated by simultaneous consideration of the stationarity conditions for transition fronts in each. Recall the basic equations for the stationary values of g_1 and g_2 are:

$$D_0 g_1 + \frac{D_1 g_1}{\left(1 + \alpha \frac{Q^2 D_1^2}{\chi_1 \Gamma^2} \frac{g_1^4}{(1 + \theta g_1)^2}\right)} = \Gamma \quad (22a)$$

$$\chi_0 g_2 + \frac{\chi_1 g_2}{\left(1 + \alpha \frac{\Gamma^2 \chi_1}{D_1^2 Q^2} \frac{g_2^4}{(1 + \eta g_2)^2}\right)} = Q. \quad (22b)$$

Here

$$\theta = (\chi_0 D_1 - D_0 \chi_1) / \chi_1 \Gamma \quad (23a)$$

$$\eta = (D_0 \chi_1 - \chi_0 D_1) / D_1 Q \quad (23b)$$

Now, ‘symmetrizing’ the definitions of the gradients via

$$f_1 = Q D_1 g_1 \quad (24a)$$

$$f_2 = \Gamma \chi_1 g_2 \quad (24b)$$

yields

$$f_2 = f_1 / (1 - \sigma f_1) \quad (24c)$$

$$f_1 = f_2 / (1 + \sigma f_2) \quad (24d)$$

where

$$\sigma = (D_0/D_1 - \chi_0/\chi_1)/Q\Gamma \equiv (\delta - \kappa)/Q\Gamma. \quad (24e)$$

It should be noted that $\sigma = 0$ is arbitrarily set ab-initio in Ref. [10], thus necessarily rendering the relation between g_1 and g_2 trivial. Using Eqn. (24), the equations for stationary f_1 and f_2 thus may be written as

$$F_1 \equiv \delta f_1 + \frac{f_1}{1 + \left(\frac{\alpha f_1^4}{\mu^2 p^2 (1 + \sigma f_1)^2} \right)} = P \quad (25a)$$

$$F_2 \equiv \kappa f_2 + \frac{f_2}{1 + \left(\frac{\alpha f_2^4}{\mu^2 p^2 (1 - \sigma f_2)^2} \right)} = P \quad (25b)$$

where $P = Q\Gamma$ and $\mu = \chi_1 D_1$. The Maxwell rule criteria for transition of f_1 and f_2 are thus easily seen to be:

$$\int_{f_{1-}}^{f_{1+}} df_1 (F_1(f_1) - P) = 0, \quad (26a)$$

$$\int_{f_{1-}}^{f_{1+}} df_2 (F_2(f_2) - P) = 0. \quad (26b)$$

Thus, the locations of the transition in density and pressure are thus seen to be different (but obviously related), since $F_1 \neq F_2$. This suggests then, that density and ion pressure (or alternatively, temperature) pedestals need neither be exactly co-located nor isomorphic! A complete analysis of the ‘dual pedestal’ problem, including the precise calculation of density and pressure transition points, requires simultaneous consideration of the regularized (i.e. with hyperdiffusion added) evolution equations for g_1 and g_2 . This is a quite difficult problem, which is beyond the scope of this paper. It will be considered in a future work.

IV. On Hysteresis

In this section we consider the possible implications of the results of the previous section for hysteresis of the system. To establish a theoretical framework for our consideration, we assume that the heating rate Q is ramped up slowly (with all other control parameters

fixed) so that the quasi-steady solutions discussed above are relevant. The conditions of Eqn. (11) can be recast in the following form

$$Q_1 < Q < Q_2 \tag{27}$$

As long as $Q < Q_1$, the system must clearly be in the L state (i.e. the lower root of Eqn. (11) should be taken). When Q exceeds Q_1 (note that this happens first at the edge, since Γ is peaked there and $Q \sim 1/\Gamma$) it is at least possible that the system *can* jump to the H-mode at all points where $Q > Q_1$. When the heating is increased beyond Q_2 , the system *must* be in the H-mode at all points where $Q > Q_2$. Now the question is where exactly in the interval $Q_1 < Q < Q_2$ does the L to H transition occur?

Hinton [19] suggested the following ansatz in response to this dilemma. He *asserted* that the system must stay at the L branch ‘as long as possible’, which implies that the transition occurs at $Q = Q_2$. On the return path, when Q is decreased from its value exceeding Q_2 down to $Q < Q_1$, the reverse transition should occur (by the same rule) at $Q = Q_1$. This clearly implies that strong hysteresis is possible in this system. Note the fact that the reverse bifurcation occurs at $Q = Q_1$ ensures that the resulting stationary profile has the ‘smoothest’ possible form (the smallest jump in g , see the S-curve in, e.g.,

Fig. 1) which apparently was one of the motivations for this rule. One significant aspect of this construction is that once the condition $Q > Q_2$ is satisfied at $x=a$, the entire volume inside the radius where $Q > Q_1$ must jump to H mode, to assure smoothness. This manifestly contradicts the declared rule of sticking to the L branch as long as possible. Most importantly, it is not clear what physical processes should be added to the basic model in order to derive this rule systematically. No justification of this ansatz was given in Refs. [9,10].

Our treatment clearly shows that the transition criterion should be linked to the Maxwell rule, instead. Formally, this rule precludes local hysteresis. However, the model under consideration is local (the flux function depends only on the local gradient), while almost any conceivable mechanism of turbulent flux suppression cannot operate strictly locally. In other words it is impossible to suppress the particle or heat flux at only one point, while leaving it unchanged (in L-mode) in the vicinity of this point. This is because the size of the suppression domain cannot be smaller than the turbulence correlation length. Therefore, the overall suppression efficiency must depend on the spatial extent of the H-mode, L_H . Obviously, L_H is closely related to the H-mode pedestal width. This clearly introduces a non-locality and an asymmetry in the bifurcation process, in the sense that once the L-H transition has occurred in an interval of finite length L_H , the entire S-curve must in turn evolve with growing L_H . Therefore, if the Maxwell rule applies, the reverse bifurcation should occur at lower Q , thus leading to hysteresis. Note in this picture, hysteresis clearly emerges as a ‘profile effect’, and is linked to profile evolution at and

behind the bifurcation front. A local model, such as that discussed here and in Refs. [9,10] is intrinsically unable to describe such hysteresis.

V. Conclusions and Discussion

In this paper, the simple S-curve bifurcation model of Hinton and Staebler has been solved analytically. This model accounts for the effects of coupled heat and particle fluxes. The principal results of this investigation are as follows.

- i.) A simple substitution allows decoupling of the density and pressure evolution equations for the specific E_r' suppression model considered.

- ii.) An analytical criterion for the coexistence of the L-phase and H-phase is derived (i.e. Eqn. (11)). This expression bounds $Q\Gamma(x)$, the required product of local heat and particle fluxes, with expressions involving the transport coefficients in L and H mode χ_1, D_1, χ_0, D_0 . This expression is considerably more general than the result of Ref. [10].

- iii.) A local transition criterion is derived. This criterion is based upon satisfying the conditions necessary for the spatial advance of one phase (i.e. L) into the other (i.e. H).

The local transition criterion is equivalent to the Maxwell construction rule. Note that distinct Maxwell rules are derived for density and pressure, indicating that the transitions for these are related but not necessarily spatially co-located. Moreover, the required value of $Q\Gamma(x)$ for each of these is, in general, different.

iv.) Detailed regularization arguments are given to support the conclusions of (iii.), above. Two regularizations are given, one in which a hyper diffusion is added and one which takes a variational approach. The results of these regularization procedures agree.

v.) The Maxwell rule criterion is shown to preclude *local* hysteresis. Thus, we argue that hysteresis is due to profile evolution effects. Thus, the space-time evolution of the S-curve must be considered in order to accurately represent hysteresis.

Several comments are in order here. First, in the spirit of ‘truth in advertising’, we reiterate that the model considered here is simplistic and limited, on account of its ‘minimalistic’ approach to the physics of $\underline{E} \times \underline{B}_0$ shear suppression of turbulence and its postulate of fixed transport coefficients χ_0, χ_1, D_0, D_1 . Also, the restriction of the theory to ‘gentle’ transitions renders it inapplicable to cases of fast power ramps, etc. Second, since the paper raises at least as many questions as it answers, the agenda for future work is indeed a lengthy one. First, the simultaneous regularization of density and pressure (or temperature) evolution must be completed. This, then, would form the foundation for a

rigorous solution of the ‘dual pedestal width’ calculation, for density and temperature. Third, this type of calculation must be implemented in tandem with transport analysis, so the unphysical modeling constraints on transport coefficients may be relaxed. Finally, the effects of noise [19] and MHD stability limits [20,21,22] should be included. The latter is a necessary element of any model of ELM phenomena [23].

Since attempts at building a simple model of the $L \rightarrow H$ transition now mark a well-trodden path, some discussion of the relation of this paper to previous work is appropriate here. We already have made extensive comparison and contrast of our results with these of Ref. [10], this paper’s closest antecedent. Hence, no further comments on Ref. [10] are given below. Regarding the ever-popular Waltz “rule” [24], based on the comparison of the linear growth rate with the local “shearing rate”, it seems clear that this approach gives no information about either bifurcation *dynamics* or H-mode layer widths and serves only as an indicator of when $\underline{E} \times \underline{B}$ shearing effects ‘might’ become important. Similarly, the heuristic approach of Onjun, *et al.* [25], serves only as a test of various ad-hoc choices for the transition layer scale and has no foundation in fundamental physics or potential for prediction. Indeed, the bi-stable sandpile model of Gruzinov, *et al.* [26,27] gives an excellent demonstration of the failure of the scaling approach of Onjun, *et al.* The empirical model of Groebner, *et al.*, [28] which links the H-mode layer or pedestal width directly to the fueling profile (i.e. neutral deposition profile) is over-simplified, in that it a-priori downplays the effects of heating power and turbulent transport, and offers no physics-based argument as to why the fueling profile should be the primary determiner of the layer scale. Indeed, indications to the contrary are discussed in Ref.

(29). The bi-stable sandpile model of Gruzinov, *et al.* [26,27] is also, in essence, a ‘one-field’ model, which does not account for heating power effects, etc. Finally, the recent and interesting ‘noisy transition’ model proposed by S.I. Itoh, *et al.* [19] is, however, limited to one field, does not treat heating and fueling, and incorporates additive noise, only. This paper does make the important point that noise-induced fluctuations can enable the transition to ‘tunnel’ thru the barrier associated with the Maxwell rule. While incomplete at present, this approach, when expanded to treat multiple fields, transport and multiplicative noise, should yield very interesting results, indeed.

While this paper is decidedly theoretical, it does, however, offer several predictions suitable for study by experimentalists. First, this theory rigorously demonstrates that *density and pressure profile development are related, but not identical*. In particular, bifurcations of density and pressure may occur at different points, and the structure of density and pressure pedestals may be different. Moreover, both pedestals can be expected to expand with increasing heating power. Pedestal expansion with increasing drive was observed in the studies of Refs. (26, 27). These predictions are quite amenable to experimental study. Indeed, there already are some indications that this is the case in experiments [30]. Unfortunately, quantitative tests must await the solution of the ‘dual regularization’ problem and the implementation of the theory in concert with transport analysis. Second, the theory predicts that hysteresis is linked to profile structure and is fundamentally non-local. This idea naturally suggests studies of the back-transition ($H \rightarrow L$ transition) in the presence of modest weak gas-puffing, which would alter H-mode density profiles and thus should impact hysteresis. Of course, the aim would be to

compare the empirical hysteresis for different H-mode edge density profiles and pedestal widths. Finally, a central result of this paper is that the criterion for advance of the barrier is given by the Maxwell construction rule. While a quantitative test of this prediction requires transport analysis, using studies of slow ramps and transitions, it may be possible to empirically construct an S-curve, as was done in Ref. [16], and see if barrier front stationarity coincides with a desposition strength which satisfies Maxwell's rule at the barrier boundary.

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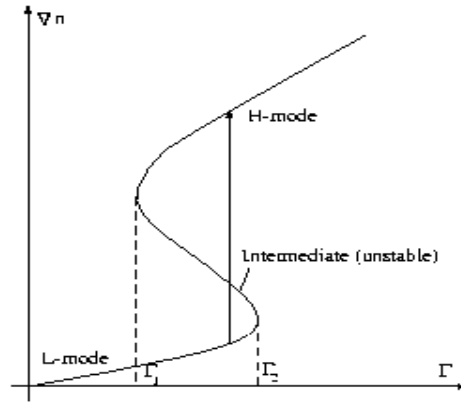


FIG. 1: Generic bifurcations diagram (S-curve) that illustrates a possible transition from the L (lower gradient) to H-mode (higher gradient) solution for a given particle deposition rate $\Gamma_1 < \Gamma < \Gamma_2$.

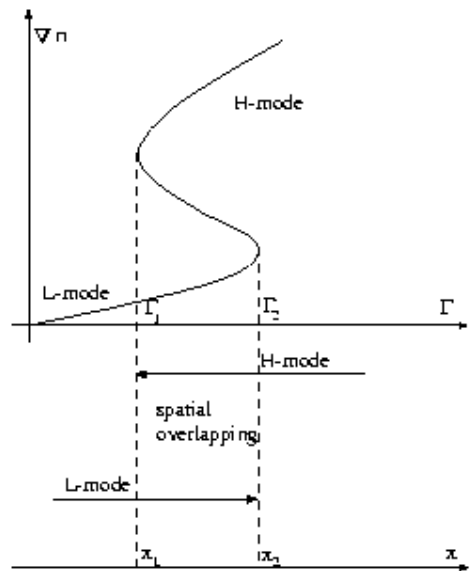


FIG. 2: The same as Fig 1 but with the underlying alignment of L and H domains that may coexist in the interval $x_1 < x < x_2$.

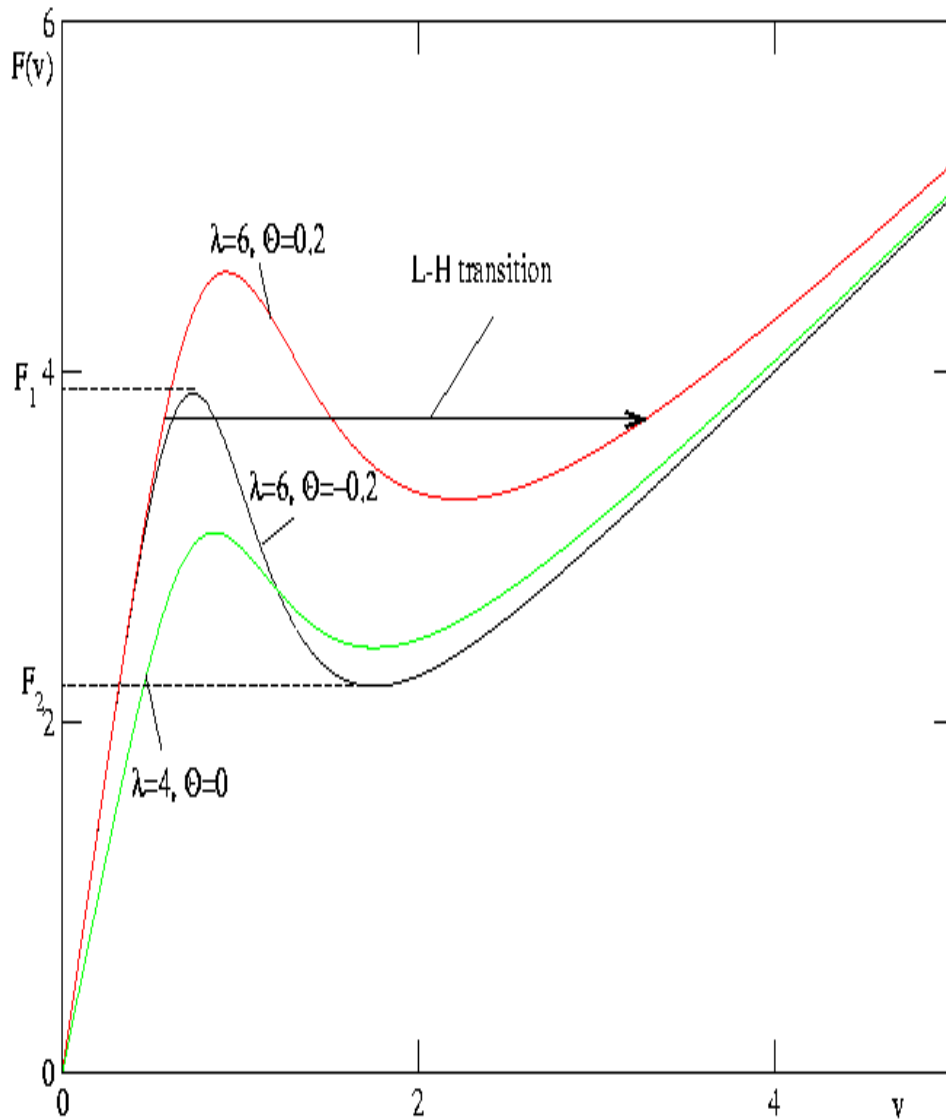


FIG. 3: Three types of behavior of the functional density $\Phi - \Gamma_g$ in Eqn. (20) and the solution of $g(x)$ that minimizes the functional Λ : (a) the L-mode part of the solution, (b) the L-H transition, and (c) the H-mode solution.

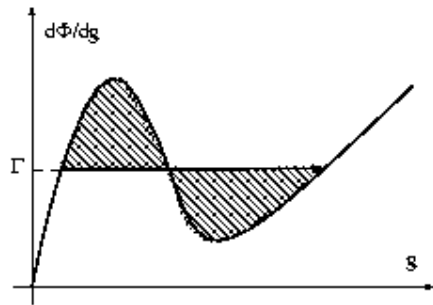


FIG. 4: Depiction Maxwell rule for the selection of transition point given analytically by Eqn. (16b).

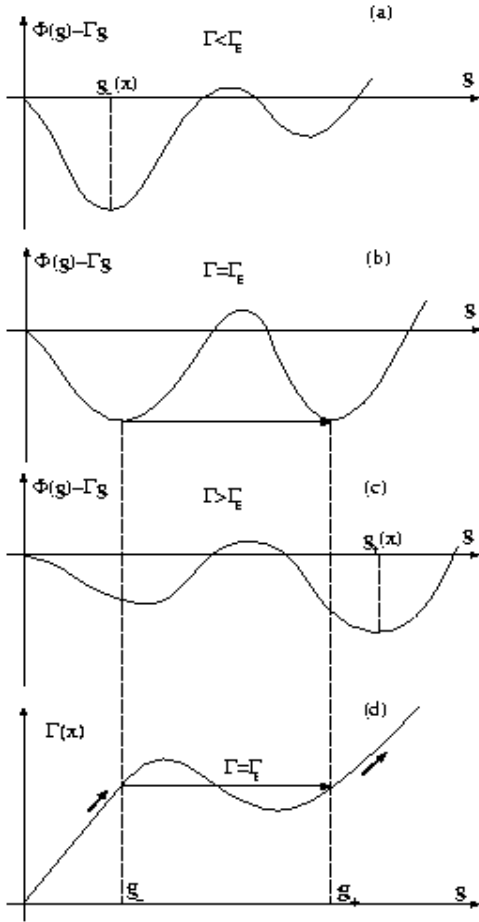


FIG. 5: Three types of behavior of the functional density $\Phi - \Gamma_g$ in Eqn. (18) and the solution $g(x)$ that minimizes the functional A: (a) the L-mode part of the solution, (b) the L-H transition and (c) the H-mode solution. The $g - \Gamma$ diagram on (d) shows the Maxwell construction corresponding to the L-H transition in (b).

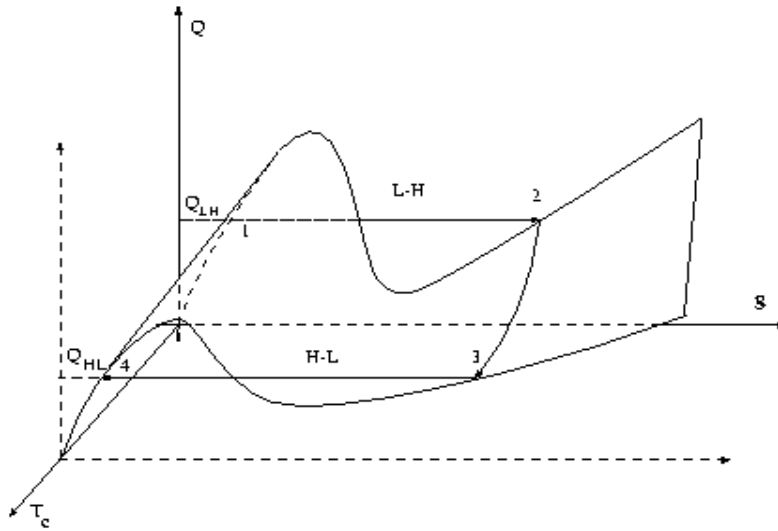


FIG. 6: Hysteresis phenomenon in L-H transition in a three dimensional parameter space that also includes the core temperature, T_c ; first, the L-H transition (1-2) occurs at a relatively low T_c since the system starts from the L mode. When the gradient g builds up at the edge, T_c increases (2-3) lowering thus the heating requirements for transitions. Then, if the heating Q decreases to its Maxwell value the reverse transition occurs at lower Q than the forward one.