Turbulent diffusion of magnetic fields in two-dimensional magnetohydrodynamic turbulence with stable stratification

Shane R. Keating* and P. H. Diamond†
Center for Astrophysics and Space Sciences,
University of California, San Diego,
9500 Gilman Drive, La Jolla, CA 92093-0424.
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We calculate the correction, due to nonlinear wave-wave interactions, to the Zeldovich relation for the turbulent diffusivity of magnetic fields $\eta_T = \eta\langle b^2\rangle/\langle B\rangle^2$, in a model of two-dimensional magnetohydrodynamic turbulence in the presence of stable stratification, where η is the molecular resistivity, b is the turbulent component of the magnetic field, and $\langle B\rangle$ is the mean component. Such a model has some relevance to hydromagnetic turbulence in stellar interiors. The significance of this correction is that, unlike the lowest order Zeldovich balance, it is independent of the molecular resistivity η, and so will not vanish in the limit of large magnetic Reynolds number, although the correction is $O(\sigma^4)$, where σ is the wave-slope, which necessarily is small. Thus, we are led to the counter-intuitive result that the presence of stable stratification can actually *increase* the vertical flux of magnetic fields relative to that in 2D MHD without stratification.

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Turbulence in a magnetized fluid presents a theoretical and conceptual challenge quite distinct from that of neutral fluids. This is because magnetohydrodynamic (MHD) turbulence is a complex dynamical system in which two fluid fields, the velocity $\mathbf{v}(\mathbf{x},t)$ and magnetic field $\mathbf{b}(\mathbf{x},t)$, evolve nonlinearly and simultaneously. Additionally, in cases where the magnetic Reynolds number $\mathrm{Rm} = \mathcal{U}\ell/\eta$ is very large (where $\mathcal{U}$ and ℓ are typical fluctuation velocities and length scales and η is the collisional resistivity), Alfvén's theorem dictates that the magnetic field will be frozen into the flow, except on small scales, where collisional resistivity allows some slippage of $\mathbf{b}$ relative to $\mathbf{v}$. The constraint imposed by this 'freezing-in' law is especially severe in the high Rm case — the purview of many astrophysical and geophysical hydromagnetic flows, where length scales can be very large and collisional resistivities very small — as the resistive diffusion rates from collisions alone are far too slow to be of any practical interest.

The freezing-in law, which is akin to, but distinct from, Kelvin's circulation theorem in neutral fluids, has some interesting implications for magnetohydrodynamic flows in two dimensions. In this case, the magnetic field may be represented by a vector potential $\mathbf{A} = A\hat{\mathbf{y}}$, say, which will be advected by the flow:

$$\partial_t A + \mathbf{v}\cdot\nabla A = \eta\nabla^2 A. \tag{1}$$

Zeldovich [1] made the observation that this has the form of a heat equation, so that the magnetic energy must ultimately decay to zero, although it may temporarily grow as a result of stretching of the field lines by the turbulent flow. Dynamo action is thus prohibited in two dimensions.

In the presence of a mean field $B_0 = \partial_z\langle A\rangle$, we expect a down-gradient diffusive flux of magnetic potential of the form $\Gamma_A = -\partial_z\langle A\rangle\eta_T = B_0\eta_T$, where η_T is the turbulent diffusivity of the magnetic potential. Multiplying (1) by A and averaging over small scales then yields, for stationary turbulence, the well-known Zeldovich theorem [17]

$$B_0^2\eta_T = \langle b^2\rangle\eta. \tag{2}$$

Equation (2), which is a direct consequence of Alfvén's freezing-in law, has important consequences for the turbulent diffusion of magnetic fields in two dimensions, as was strikingly illustrated by the seminal study of Cattaneo and Vainshtein [2]. Employing a combination of physical argument and numerical calculation, they demonstrated that the turbulent resistivity η_T (i.e. the turbulent diffusivity of magnetic flux) is suppressed below the kinematic value $\eta_{\mathrm{kin}} = \langle v^2\rangle\tau_c$ by an Rm-dependent factor: $\eta_T = \eta_{\mathrm{kin}}(1+\mathrm{Rm}\langle B\rangle^2/\langle v^2\rangle)^{-1}$, where the mean field $\langle B\rangle$ is measured in units of the Alfvén velocity. This expression was later derived using a quasi-linear closure by Gruzinov & Diamond [3]. For high Rm flows this suppression, or 'quenching', can be significant indeed, even for weakly magnetized fluids satisfying $\langle B\rangle^2 > \mathrm{Rm}^{-1}\langle v^2\rangle$. The result of Cattaneo and Vainshtein engendered considerable debate, particularly when the result was extended to the α-effect in three dimensions so a similar suppression was found to act on dynamo action [4–7]. In view of the enormous values of Rm found in astrophysical flows (10^9 in stars, and even greater for the galaxy), this suppression is sometimes termed 'catastrophic quenching', and places a serious limit on the observable flux production in cosmical dynamos.

In this Letter, we consider an extension to the theory of turbulent diffusion of magnetic fields in two-dimensional MHD turbulence to include the effect of an imposed stable stratification [8]. This system constitutes a simple model relevant to the theory of of MHD turbulence in a convectively stable stellar interior. The radiation zone of the Sun is strongly stably stratified with a Brunt-Väisälä frequency of about $2.5 \times 10^{-3}\mathrm{s}^{-1}$. At the interface between the convection zone and the radiation zone, in a thin layer known as the tachocline, solar dynamo activity is thought to give rise to a strong toroidal field of about 10^4–10^5 gauss [9–11], so that plasma motions here are likely to be severely constrained by radial stratification. Turbulent coefficients, such as turbulent resistivity and viscosity, are of interest in considerations of tachocline structure [12, 13].

Let the motion be confined to the xz-plane, so that the velocity and total magnetic field are described by a stream function $\phi(\mathbf{x},t)$ and magnetic potential $A(\mathbf{x},t)$ such that $\mathbf{v} = \nabla\phi\times\hat{\mathbf{y}}$, and $\mathbf{B} = \nabla A\times\hat{\mathbf{y}}$. The usual governing equations for incompressible two-dimensional MHD are modified by the appearance of a buoyancy term in the Navier-Stokes equation and an additional equation governing the density $\rho(\mathbf{x},t)$:

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla P_{\text{eff}} + \frac{1}{4\pi}\mathbf{B}\cdot\nabla\mathbf{B} - \rho g\hat{\mathbf{z}} + \rho\nu\nabla^2\mathbf{v}, \qquad (3)$$

$$\frac{\mathrm{D}A}{\mathrm{D}t} = \eta\nabla^2 A, \quad \frac{\mathrm{D}\rho}{\mathrm{D}t} = \mathcal{D}\nabla^2\rho, \qquad (4)$$

where $\mathrm{D}/\mathrm{D}t = \partial_t + \mathbf{v}\cdot\nabla$ is the usual advective derivative, and ν, η and $\mathcal{D}$ are the molecular viscosity, resistivity and mass diffusivity, respectively. Both thermal and magnetic pressure are contained in P_{eff}. Gravity $\mathbf{g} = -g\hat{\mathbf{z}}$ is pointed in the negative z-direction.

Separating the magnetic potential into mean and fluctuating components $A = \langle A\rangle(z) + \tilde{A}(\mathbf{x},t)$ implies that the mean and fluctuating components of the magnetic field are $\langle\mathbf{B}\rangle = B_0\hat{\mathbf{x}} = -\partial_z\langle A\rangle\hat{\mathbf{x}}$ and $\mathbf{b} = \nabla\tilde{A}\times\hat{\mathbf{y}}$, and will again be measured in units of the Alfvén velocity. In addition, vertically stable stratification is assumed, so that the density field can also be separated into a mean and fluctuating component $\rho = \langle\rho\rangle(z) + \tilde{\rho}(\mathbf{x},t)$, where $\langle\rho\rangle$ has a negative gradient in the z-direction, and we may define the usual Brunt-Väisälä frequency N associated stable stratification as $N^2 = -g\partial_z\ln\langle\rho\rangle \geq 0$.

Consistent with a Boussinesq approximation, it is assumed that the fluctuation in density $\tilde{\rho}$ appears only in the buoyancy term. Therefore, taking the curl of (3) yields the following equations for the fluctuating fields (dropping tildes where there is no ambiguity):

$$\partial_t\omega - \frac{\mathrm{d}\langle A\rangle}{\mathrm{d}z}\partial_x\nabla^2 A - \frac{g}{\langle\rho\rangle}\partial_x\rho = \mathbf{v}\cdot\nabla\omega - \mathbf{b}\cdot\nabla j + \nu\nabla^2\omega, \qquad (5)$$

$$\partial_t A + \frac{\mathrm{d}\langle A\rangle}{\mathrm{d}z}v_z = -\mathbf{v}\cdot\nabla A + \eta\nabla^2 A, \qquad (6)$$

$$\partial_t\rho + \frac{\mathrm{d}\langle\rho\rangle}{\mathrm{d}z}v_z = -\mathbf{v}\cdot\nabla\rho + \mathcal{D}\nabla^2\rho. \qquad (7)$$

where $\omega = -\nabla^2\phi$ and $j = -\nabla^2 A$ are the vorticity and current density in the y-direction.

The addition of buoyancy forces to the equations of motion has the effect of converting large-scale eddies into dispersive magneto-internal waves, as can be seen from the dispersion relation obtained from the linearized dissipationless equations of motion, $\Omega_{\mathbf{k}}^2 = \Omega_{\mathbf{k}}^{\mathrm{AW}2} + \Omega_{\mathbf{k}}^{\mathrm{IW}2}$, where $\Omega_{\mathbf{k}}^{\mathrm{AW}} = B_0 k_x$ is the Alfvén wave frequency and $\Omega_{\mathbf{k}}^{\mathrm{IW}} = Nk_x/|\mathbf{k}|$ is the frequency of an internal gravity wave. The linear modes, therefore, are hybrid, 'magneto-internal' waves. For our purposes, the most pertinent property of these waves is that, on small scales, they behave like Alfvén waves and are non-dispersive, whereas on large scales they are more like pure internal gravity waves, which are dispersive. In addition, those waves with a wavelength above a threshold length scale (specifically, those scales for which the wave-slope $k\tilde{\varepsilon}$ is less than unity, where $\tilde{\varepsilon}$ is a fluctuation displacement element) will interact weakly, transferring energy among resonant modes. By contrast, wave interactions on small scales are washed out by turbulent de-correlation before they can interact resonantly. With some additional well-known assumptions — briefly, the existence of a broad spectrum of weakly interacting dispersive waves — the turbulence on large scales can therefore be described by wave turbulence theory [8, 14, 15].

Wave turbulence theory has the advantage of possessing a source of small-scale irreversibility which is present even in the case of $\eta \to 0$, that is, $\mathrm{Rm} \to 0$: three-wave resonances, which are present even in the dissipationless limit, via the Landau pole prescription, and appear in the theory as $\delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \omega_{\mathbf{k}+\mathbf{k}'})$. The wave-triads identified by this resonance condition are those which make a secular contribution to irreversible energy transfer among interacting modes. As we shall demonstrate, the *spectral transfer of energy* among resonant modes also gives rise to the *spatial transport of magnetic potential.*

The vertical flux of the magnetic potential A is given by $\Gamma_A = \langle v_z\delta A\rangle + \langle A\delta v_z\rangle$, where angle brackets denote a spatial average and δv_z and δA represent the response of the fluid and the field to wave interactions. Within the region of wave-number space in which wave turbulence theory is valid, the linear fields v_z, A and ρ are simply due to wave oscillations, and can be expressed in terms of the vertical wave-displacement, ϵ, defined by $v_z = \partial_t\epsilon$. Likewise, neglecting nonlinear and dissipative terms from (6) and (7) gives, for the linear fields, $A = -\partial_z\langle A\rangle\epsilon$ and $\rho = -\partial_z\langle\rho\rangle\epsilon$. Substituting for these linear fields in Γ_A gives

$$\Gamma_A = \langle\partial_t\epsilon\delta A\rangle - \langle\epsilon\partial_z\langle A\rangle\delta v_z\rangle. \qquad (8)$$

For stationary turbulence, the time derivative of averaged quantities will vanish, so that (8) can be written as

$$\Gamma_A = -\langle\epsilon\left(\partial_t\delta A + \partial_z\langle A\rangle\delta v_z\right)\rangle. \qquad (9)$$

The expression in round brackets in (9) is, from equation (6), simply $-\delta\left(\mathbf{v}\cdot\nabla A\right)+\eta\nabla^2\delta A$, so that,

$$\Gamma_A=\langle\epsilon\delta\mathbf{v}\cdot\nabla A\rangle+\langle\epsilon\mathbf{v}\cdot\nabla\delta A\rangle-\eta\langle\epsilon\nabla^2\delta A\rangle. \quad (10)$$

The first expression on the right-hand side of (10) is, by virtue of the expression for the linear field A, proportional to $\langle\epsilon\delta\mathbf{v}\cdot\nabla\epsilon\rangle=\langle\nabla\cdot\left(\delta\mathbf{v}\epsilon^2\right)\rangle-\langle\epsilon^2\nabla\cdot\delta\mathbf{v}\rangle$ and will vanish for periodic boundary conditions and incompressibility of $\delta\mathbf{v}$. (Again, ϵ is the *vertical* wave displacement.) The total vertical flux of magnetic potential is then

$$\Gamma_A=\Gamma_c+\Gamma_{ww}, \quad (11)$$

where $\Gamma_c=-\eta\langle\epsilon\nabla^2\delta A\rangle$ is the flux driven by molecular collisions, and $\Gamma_{ww}=\langle\epsilon\mathbf{v}\cdot\nabla\delta A\rangle$ is the flux driven by nonlinear wave-wave interactions.

As required for the validity of wave turbulence theory, the wave-slope $k\epsilon$ must be strictly less than unity, so that the response δA can be expanded in powers of $k\epsilon$, $\delta A=\delta A^{(1)}+\delta A^{(2)}+\delta A^{(3)}+\cdots$, where $\delta A^{(1)}=A$ is due to wave oscillations and the higher order terms are due to wave interactions. Therefore, the lowest order contribution to Γ_A comes from the collisional flux, $\Gamma_A^{(2)}=\eta\langle b^2\rangle/B_0$. Assuming a Fickian diffusion $\Gamma_A=-\eta_T\partial_z\langle A\rangle$, we recover the Zeldovich theorem $\eta_T^{(2)}=\eta\langle b^2\rangle/B_0^2$.

To calculate the correction to the Zeldovich theorem arising from wave-wave interactions, we express Γ_{ww} in terms of Fourier components:

$$\Gamma_{ww}=\mathrm{Re}\sum_{\Delta}\left(\mathbf{k}'\cdot\mathbf{k}''\times\hat{\mathbf{y}}\right)\epsilon_{\mathbf{k}'\omega'}\phi_{\mathbf{k}''\omega''}\delta A_{\mathbf{k}\omega}, \quad (12)$$

where the summation is over Δ, the set of all wave triads $(\mathbf{k},\omega)$, $(\mathbf{k}',\omega')$, $(\mathbf{k}'',\omega'')$ satisfying the resonance conditions $\mathbf{k}+\mathbf{k}'+\mathbf{k}''=\mathbf{0}$, $\omega+\omega'+\omega''=0$.

We shall assume that each field has associated with it a random phase (the random phase approximation, or RPA): this has the consequence that all odd cumulants vanish. Thus, the lowest-order contribution to Γ_{ww} is

$$\Gamma_{ww}^{(4)}=\mathrm{Re}\sum_{\Delta}\left(\mathbf{k}'\cdot\mathbf{k}''\times\hat{\mathbf{y}}\right)\epsilon_{\mathbf{k}'\omega'}\phi_{\mathbf{k}''\omega''}\delta A^{(2)}_{\mathbf{k}\omega}. \quad (13)$$

It is worth noting that (13) is independent of $\delta\rho^{(2)}$ or $\delta\phi^{(2)}$, so we need calculate *only* $\delta A^{(2)}$.

In Fourier space, the equations of motion (3) and (4) can be written in the form $\partial_t\mathbf{u}_\mathbf{k}=\mathrm{i}\mathcal{L}_\mathbf{k}\mathbf{u}_\mathbf{k}+\mathbf{N}_\mathbf{k}\left(\mathbf{u},\,\mathbf{u}\right)$, where $\mathbf{u}_\mathbf{k}$ is the vector of dynamical fields, $\mathcal{L}_\mathbf{k}$ is the linear operator (a matrix), and $\mathbf{N}_\mathbf{k}$ are the quadratic nonlinearities. The second-order responses $\delta\mathbf{u}^{(2)}$ then satisfy $\partial_t\delta\mathbf{u}^{(2)}_\mathbf{k}=\mathrm{i}\mathcal{L}_\mathbf{k}\delta\mathbf{u}^{(2)}_\mathbf{k}+\mathbf{N}_\mathbf{k}\left(\mathbf{u},\,\mathbf{u}\right)$, with the formal solution

$$\delta\mathbf{u}^{(2)}_{\mathbf{k}\omega}=\frac{\mathrm{i}}{\omega+\mathcal{L}_\mathbf{k}+\mathrm{i}0^+}\mathbf{N}_{\mathbf{k}\omega}\left(\mathbf{u},\,\mathbf{u}\right), \quad (14)$$

where the presence of 0^+ ensures causality. The RPA then implies that the $(\mathbf{k},\omega)$ mode is driven by the beating of the $(\mathbf{k}',\omega')$ and $(\mathbf{k}'',\omega'')$ modes:

$$\mathbf{N}_{\mathbf{k}\omega}=\mathbf{u}_{\mathbf{k}'\omega'}{}^{*}\mathcal{N}_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\mathbf{u}_{\mathbf{k}''\omega''}{}^{*}, \quad (15)$$

where $\mathcal{N}$ is the interaction tensor. Finally, expressing the linear fields $\mathbf{u}$ in terms of the displacement ϵ yields

$$\mathbf{N}_{\mathbf{k}\omega}=\mathbf{n}_{\substack{\mathbf{k}',\mathbf{k}''\\ \omega',\omega''}}\epsilon_{\mathbf{k}'\omega'}\epsilon_{\mathbf{k}''\omega''}. \quad (16)$$

Substituting (14) into the expression for $\Gamma^{(4)}_{ww}$ then gives the lowest-order contribution to the wave-interaction driven flux. The exact form of $\Gamma^{(4)}_{ww}$ depends upon the linear operator $\mathcal{L}$ and the interaction tensor $\mathcal{N}$; for stratified MHD it is $\Gamma^{(4)}_{ww}=-\partial_z\langle A\rangle\eta^{(4)}_{ww}$, with

$$\eta^{(4)}_{ww}=\frac{1}{8}\sum_{\Delta}g_{\mathbf{k}',\mathbf{k}''}\left(\mathcal{C}^+\theta^++\mathcal{C}^-\theta^-\right)|\sigma_{\mathbf{k}'\omega'}|^2|\sigma_{\mathbf{k}''\omega''}|^2, \quad (17)$$

where $\sigma_{\mathbf{k}\omega}=|\mathbf{k}|\epsilon_{\mathbf{k}\omega}$ is the wave-slope of the $(\mathbf{k},\omega)$ mode, $g_{\mathbf{k}',\mathbf{k}''}=(\hat{\mathbf{e}}_{\mathbf{k}'}\cdot\hat{\mathbf{e}}_{\mathbf{k}''}\times\hat{\mathbf{y}})^2$ is a geometrical factor, and the coupling coefficients $\mathcal{C}^\pm$ are

$$\begin{aligned}\mathcal{C}^\pm &= \frac{k_x}{\Omega_\mathbf{k}}\frac{k'^2-k''^2}{k^2}\left(\frac{\omega'}{k'_x}-\frac{\omega''}{k''_x}\right)\left(\frac{\omega'}{k'_x}\frac{\omega''}{k''_x}-B_0^2\right)\\ &\quad\pm\left(\frac{\omega'}{k'_x}-\frac{\omega''}{k''_x}\right)^2 \end{aligned} \quad (18)$$

The response times $\theta^\pm=\delta(\omega\pm\Omega_\mathbf{k})$ come from taking the real part of $\mathrm{i}(\omega\pm\Omega_\mathbf{k}+\mathrm{i}0^+)^{-1}$.

The lowest order correction to the Zeldovich theorem is then

$$\eta_T=\eta\frac{\langle b^2\rangle}{B_0^2}+\eta^{(4)}_{ww}+\cdots \quad (19)$$

where the ellipsis denotes terms of order $\eta\sigma^2$, σ^6 and higher. The asymptotic behavior of the two terms on the right of (19) is set by two dimensionless parameters: Rm (through η) and $k\tilde{\varepsilon}$. For sufficiently high Rm, the second-order term, which is proportional to Rm^{-1}, is dominated by the fourth-order term, which has no dependence on Rm.

Crucially, Rm and $k\epsilon$ are *independent* asymptotic parameters, measuring, as they do, the ratio of different dimensional quantities. This is most clearly demonstrated in terms of time-scales: Rm is the ratio of the diffusive time-scale τ_D to the advective time-scale τ_{NL}, which is set by the nonlinearity. The timescale τ_{NL} also describes the rate of nonlinear steepening of waves, and the ratio of the period $1/\omega_k$ of a wave with wavelength k to τ_{NL} is simply the wave-slope $k\epsilon$. Unlike Rm, which defined relative to some reference scale, $k\epsilon$ must be determined scale-by-scale. Therefore, η_T is dominated by wave-wave interactions in the dual asymptotic limit $k\epsilon\ll 1\ll\mathrm{Rm}$, or, ${\omega_k}^{-1}\ll\tau_{NL}\ll\tau_D$.

Finally, it is worthwhile pointing out that the correction η_{ww}, appearing in (19), can be reconciled with the Zeldovich theorem (2) by noting that the assumption of stationarity, which was a key element in the derivation of (2), is a flawed one. Indeed, as is well known, no stationary state exists for magnetic fields in two dimensions. In the presence of wave interactions then, a new time-scale T is introduced via the response functions $\theta^{\pm}$. This time-scale modifies the Zeldovich balance (2) to $\partial_T\langle A^2\rangle + \eta_T B_0^2 = \eta\langle b^2\rangle$. Therefore, equation (19) can be reconciled with the Zeldovich theorem by identifying η_{ww} with $-\partial_T\langle A^2\rangle/B_0^2$.

Conclusion — In the presence of stable stratification, the Zeldovich theorem $\eta_T = \eta\langle b^2\rangle/B_0^2$ is modified by interacting magneto-internal waves, which introduce a new time-scale associated with the slow transfer of energy among resonant wave triads. We have calculated the lowest-order contribution to the flux arising from such wave-wave interactions and have shown that, unlike the flux driven by molecular collisions, it is independent of the molecular resistivity η and hence the magnetic Reynolds number Rm, although it is still limited by the conditions of wave turbulence theory. In the limit $\eta \to 0$ ($\mathrm{Rm} \to \infty$) the flux driven by wave interactions will remain finite (but small in $k\epsilon < 1$), while the collisional flux will be strongly quenched. Thus we are led to the surprising and counter-intuitive conclusion that, all other factors (such as forcing and dissipation) being equal, the addition of buoyancy to the already tightly constrained system of homogeneous high Rm two-dimensional MHD can actually *increase* the transport of mean magnetic potential. Wave-wave interactions, therefore, place a significant limit on the theory of 'catastrophic' resistivity quenching in astrophysical magnetofluids.

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* Electronic address: skeating@physics.ucsd.edu

† Electronic address: phd@mamacass.edu

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[17] A distinction must be made between the "Zeldovich theorem", which follows directly from Alfvén's freezing-in law, and the so-called "Zeldovich relation" [1], which estimates the maximum value the small-scale field can grow to from a large-scale seed field (see [16]). This nomenclature has been used by several authors, and we follow that convention here.