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β-Plane MHD Turbulence and Dissipation in the Solar Tachocline

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Abstract

The physical processes causing the turbulent dissipation and mixing of momentum and magnetic fields in the solar tachocline are discussed in the context of a simple model of 2D MHD turbulence on a β-plane. The mean turbulent resistivity and viscosity for this model are calculated. Special attention is given to the enhanced dynamical memory induced by small scale magnetic fields and to the effects of magnetic fluctuations on nonlinear energy transfer. The analogue of the Rhines scale for β-plane MHD is identified. The implications of the results for models of the solar tachocline structure are discussed.

9.1 Introduction

The tachocline is a thin, stably stratified layer of the solar interior situated in the radiative zone, immediately below the convection zone (Miesch 2005; Tobias 2005). This layer connects the latitudinal differential rotation of the solar convection zone to the expected solid body rotation of the solar interior (Schou *et al.* 1998; see also Chapter 3 by Christensen-Dalsgaard & Thompson). Thus, flows in the tachocline are sheared (both poloidally and radially) with the predominant structure being that of a radially sheared toroidal flow. The stratification of the tachocline is strongly stable (with Richardson number $Ri \gg 1$), and the magnetic field strength is significant, though magnetic pressure is still much smaller than thermal pressure, consistent with hydrostatic equilibrium, i.e. $B^2/8\pi \ll p$.

In addition to its intrinsic interest, the tachocline has received considerable attention recently on account of its pivotal role in the proposed interface dynamo of the Sun (Parker 1993; see also Chapter 13 by Tobias & Weiss). Interest in the interface dynamo has been sparked by the many fundamental questions recently raised concerning the mean field $\alpha\omega$ theory of the solar dynamo and its traditional constituents, the alpha (α) and turbulent diffusivity (η_T) effects.† In this regard, possible quenching of η_T and α in inverse proportion to the product of magnetic Reynolds number (Rm) and mean magnetic field intensity (B_0^2), (i.e. $\sim (1+RmB_0^2)^{-1}$), is a particularly strong motivation to consider alternative dynamo scenarios (Cattaneo & Vainshtein 1991; Gruzinov & Diamond 1994; Diamond, Hughes & Kim 2005a). The interface dynamo concept proposes an escape from quenching by separating the location of cyclonic turbulence (which drives the α-effect) from the site of shearing (ω-effect) and the consequent strong magnetic field build-up. In the interface dynamo, weak poloidal fields are amplified by α in the convection zone, then transported to the tachocline by either turbulent diffusion or entrainment by convectively overshooting plumes (a process referred to as 'magnetic flux pumping'; Tobias *et al.* 2001). Once in the tachocline, the magnetic field is amplified by flow shearing to form strong toroidal loops. A cartoon of the interface dynamo scenario is shown in Figure 9.1. The toroidal field remains stored below the convection zone, until it erupts upward into the convection zone, and eventually into the solar atmosphere (Parker 1966; Hughes 1991). Also, the nature of field amplification in the interface dynamo is such that the overall sensitivity to the value of α is reduced. In the interface scenario α is required only to convert toroidal field to poloidal, while toroidal field is actually amplified by tachocline shear. This state brings to mind the familiar contrast between α^2 and $\alpha\omega$ dynamos. In α^2 dynamos, the field growth rate $\gamma \sim \alpha$ while for $\alpha\omega$ dynamos $\gamma \sim \sqrt{\alpha\Omega}$ with the reduced sensitivity to α evident. Given the many attractive features of the interface dynamo, it is no stretch to say that understanding the tachocline is an important prerequisite for a theory of the solar dynamo, solar cycle, etc.

Perhaps the most basic and important questions concerning the tachocline are:

(i) why does it exist?
(ii) where is it located and why is it so sharply localized?

Regarding existence, the tachocline is formed by the penetration of a shear into the stably stratified radiation zone. This process of sheared flow penetration is ultimately driven by solar spin-down, i.e. the loss of angular momentum from the Sun on account of its coupling to the outgoing, rotating, solar wind (Bretherton & Spiegel 1968; Spiegel & Zahn 1992). Spin-down in

† Note that the latter is often denoted by β. To avoid confusion in this chapter we use η_T.

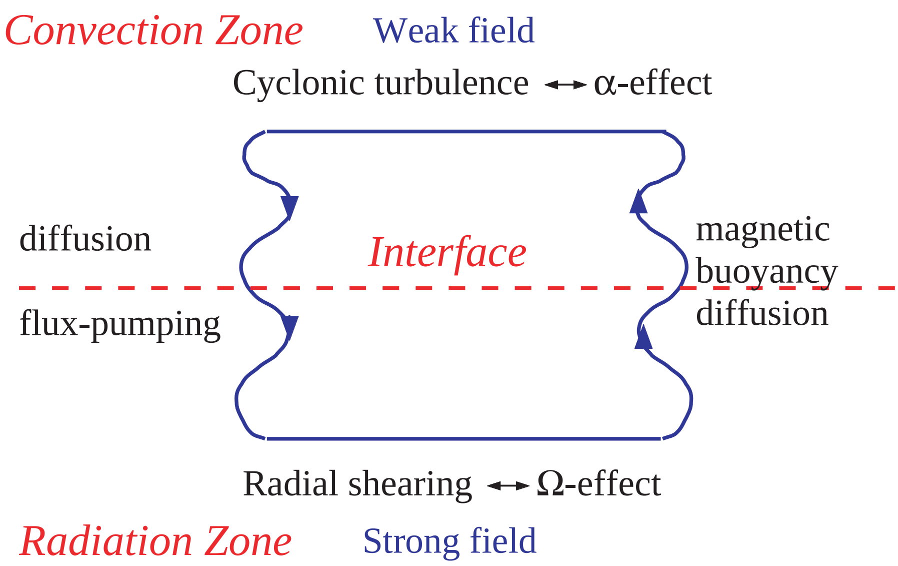

Fig. 9.1. A schematic of the interface dynamo cycle, which operates near the boundary of the convection and radiation zones. In this cartoon the tachocline sits just below the convection zone. Magnetic field is amplified by an α-effect in the convection zone, 'pumped' into the tachocline by convective overshoot, amplified by shearing in the tachocline, and returned to the convection zone by magnetic buoyancy.

turn generates meridional circulation cells, which drive the inward penetration of shear. Alternatively put, approximate *thermal wind balance* tightly links radiative heat transport-driven baroclinic torques to the vertical variation of the centrifugal force (Mestel 1999). Any small deviation from exact thermal wind balance necessarily implies meridional circulation, which must exist to balance the torque budget. Thus, it may be said that meridional cells are spawned by the competition between stratification (N^2) and rotation (Ω^2), in the sense that these two effects directly compete against one another in the thermal wind balance. This process of meridional flow-driven 'burrowing' is sometimes referred to as *gyroscopic pumping* (McIntyre 2000, 2003), and is closely related to the well-known mechanism of the Eddington-Sweet circulation in convectively stable stellar interiors (Eddington 1926; Sweet 1950). Indeed, the basic time scale of tachocline penetration is the Eddington-Sweet time scale $\tau_{ES} = (N/2\Omega)^2 r_0^2/\kappa$, where r_0 is the radius of the tachocline boundary, κ is the thermal diffusivity, N is the Brunt-Väisälä frequency and Ω is the rotation frequency.

The second question above, i.e. what limits or localizes the tachocline, is the more challenging one, by far. Radial mixing is ineffective, on account of the severe limitation imposed on it by the strong stable stratification in the solar radiation zone. It would just slightly enhance the Eddington-Sweet penetration rate, so making little or no difference in the outcome of that theory. Thus, one must turn to turbulent viscous mixing, as in the Spiegel-Zahn (S.-Z.) scenario (Spiegel & Zahn 1992), in which tachocline burrowing is balanced by 'horizontal' turbulent momentum transport, or to magnetic field effects, as in the Gough-McIntyre (G.-McI.) scenario (Gough & McIntyre 1998). In the latter scenario, tachocline penetration is opposed by a hypothetical fossil dipolar magnetic field in the solar radiation zone. The fossil dipole field is separated from the tachocline and convection zone by a magnetic separatrix or 'tachopause', at which nonlinear dissipative MHD processes (i.e. magnetic reconnection) are both crucially important and excruciatingly difficult to calculate. One key element in determining the tachocline thickness according to the G.-McI. model is the balance of shearing of poloidal fields with resistive dissipation of toroidal magnetic fields. This magnetic dissipation can be either resistive (i.e. related to radial diffusion, as actually assumed by G.-McI.) or turbulent, and so related to poloidal transport and mixing of magnetic fields. Thus, it is interesting to note *that in both tachocline formation scenarios (i.e. S.-Z. and G.-McI.), turbulent transport and dissipation play central roles.* Finally, we note here that other tachocline models exist but are beyond the scope of this discussion (see Chapter 7 in this volume by Garaud). For an alternative magnetic model see Rüdiger & Kitchatinov (1977).

Tachocline turbulence is quasi-geostrophic turbulence in a spherical shell, and is excited primarily by forcing due to convective overshoot. The forcing may also be thought of as a surrogate for the input of energy to smaller scales as a result of large scale instability. As we argue below, tachocline turbulence almost certainly has a strong magnetic component and thus should be thought of as quasi-geostrophic MHD turbulence, the *simplest* incarnation of which is β-plane MHD turbulence. Here, we consider the β-plane model, rather than a spherical surface or spherical shell model, for reasons of simplicity. Predictive understanding of turbulent transport and dissipation of *both* momentum (closely related to vorticity in 2D) and magnetic fields in β-plane MHD turbulence is necessary in order to construct tachocline formation models. *Such turbulent transport provides the key element of dissipation, which limits or offsets the meridional cell-driven 'burrowing'* (McIntyre 2003). This paper discusses the physics of transport and dissipation in β-plane MHD. Both momentum and magnetic field transport are considered.

There are at least three specific reasons why understanding β-plane MHD turbulence is an interesting and challenging task. These are:

(i) the simultaneous presence and coexistence of eddies, Rossby waves and Alfvén waves at different scales;
(ii) the freezing of magnetic potential and field into the fluid at high magnetic Reynolds number Rm. 'Freezing in' has been shown to severely limit turbulent diffusion of magnetic fields in 2D;
(iii) the tendency of even 2D turbulence to stretch magnetic fields and thus to 'Alfvénize' the turbulence, producing a high intensity spectrum of small scale magnetic fields. Also, in 2D MHD, energy forward cascades, rather than inverse cascades as in a 2D fluid, once the magnetic field intensity exceeds a weak minimal level (Kraichnan 1965; Pouquet 1978).

These three observations in turn suggest that:

(i) as in the case of ordinary geostrophic turbulence, a scale emerges in β-plane MHD turbulence which demarks the boundary between an 'eddy turbulence' range of scales and a 'wave turbulence' range. *We shall show that in β-plane MHD, this analogue of the well-known Rhines scale (Rhines 1975) can be Rm dependent*;
(ii) for scales $\ell < \ell_{RM}$, where ℓ_{RM} is the β-plane MHD 'Rhines scale', the dynamics is essentially that of *2D MHD turbulence*. So, turbulence tends to 'Alfvénize', thus enhancing memory and quenching turbulent diffusion and dissipation of both momentum and field. Moreover, energy *forward* cascades, even in 2D MHD;
(iii) for scales $\ell > \ell_{RM}$, the dynamics is that of a gas of Rossby waves. In particular, turbulent transport is controlled by *wave interaction*, so the simplified conventional wisdom about 2D MHD turbulence is not applicable.

The upshot of all this is that the actual *dynamics* of turbulent mixing of momentum and magnetic field in the tachocline is quite unclear, and that horizontal turbulent viscosity (ν_h) and resistivity (η_h) are either quenched or significantly reduced! This discussion indicates the need to seriously consider the micro-physics of turbulent transport in the tachocline environment when constructing models of tachocline formation.

The remainder of this paper is organized as follows. Section 9.2 introduces the model and discusses some basic aspects of β-plane MHD, including the

Table 9.1. *Elements of Tachocline Formation Scenarios*

Element	*Spiegel-Zahn Model*	*Gough-McIntyre Model*
Drive of Tachocline Formation	Spin Down, Meridional Circulation Cells	Spin Down, Meridional Circulation Cells
Tachocline Limiter	Horizontal Viscosity and Momentum Mixing	Fossil Poloidal Magnetic Field in Radiative Core
Turbulent Dissipation Mechanism	Turbulent *viscosity* in quasi-geostrophic fluid or MHD	Turbulent *resistivity* in quasi-geostrophic 2D MHD
Critical Balance	$\nu_h \nabla_h^2 \mathbf{v}$ *vs.* 'burrowing' of tachocline	$\eta \partial_r^2 + \eta_h \partial_h^2$ *vs.* shearing of B_p into B_ϕ
Critical Issues in Turbulence Physics	1) cascade direction 2) Rhines scale in β-plane MHD 3) Momentum Transport	1) turbulent resistivity, quenching of η_T in β-plane MHD 2) $\langle A^2 \rangle$ spectral transport direction

important Zeldovich theorem. The effective Rhines scale for β-plane MHD is discussed in §9.3. In particular, we show that in β-plane MHD at high Rm, the effective Rhines scales ℓ_{RM} increases with Rm. Spectral transfer and turbulent dissipation are discussed in §9.4. The eddy and Alfvén wave forward cascade range at scales $\ell < \ell_{RM}$ and the Rossby wave dominated range at $\ell > \ell_{RM}$ are dealt with separately. We show that for $\ell < \ell_{RM}$, both ν_h and η_h are significantly reduced by the effects of small scale magnetic fields. We also discuss the effects of Rossby wave interactions on turbulent transport at larger scales. Section 9.5 discusses the implications of these results for the tachocline structure formation scenarios.

9.2 Some Basic Aspects of 2D MHD Turbulence on a β-Plane

Though virtually all previous studies of turbulence and turbulent transport in the tachocline have been in the context of neutral fluid models, tachocline turbulence is very likely *MHD turbulence*, or turbulence with a substantial magnetic component. This assumption is natural, given the strong toroidal magnetic field of the tachocline *and* the presence of magnetic field sources both above and below the tachocline. Specifically, the tachocline magnetic field is fueled from above by overshooting plumes, which originate in the convection zone and which entrain convection zone magnetic fields while they fall into the stably stratified tachocline below the convection zone. Likewise, small elements or loops of the fossil field thought to reside in the solar radiation zone (as in the G.-McI. scenario) may enter the tachocline following reconnection events, which occur at the tachopause, and which thus fuel the tachocline magnetic field from *below*. Thus, the readily available sources of magnetic field as well as the stable stratification and apparent minute thickness of the layer suggest that turbulence in the tachocline is *'shellular' MHD turbulence*, which is two-dimensional in character. Here we make the simplest of approximations and neglect the thickness of the shell, thus taking the dynamics to be 2D. Since shellular turbulence in a layer of finite thickness can exhibit complex vertical couplings (as in multi-layer models; Pedlosky 1987), further simplification is desirable. Thus, we focus on the *absolutely minimal model* of such shellular MHD turbulence, *namely two dimensional MHD turbulence on a β-plane* (Bracco *et al.* 1998). This system includes constituents of both geostrophic turbulence (i.e. Rossby waves, vortices) and 2D MHD turbulence (i.e. Alfvén waves etc). Despite the simplicity of this minimalist model, developing a theory of β-plane MHD turbulence is still useful for tachocline modeling, since such a theory can constrain and elucidate the physics of the turbulent transport and dissipation coefficients (i.e. turbulent viscosity and resistivity) which (partially) determine the structure and the thickness of the tachocline in either the S.-Z. or the G.-McI. scenario. In particular, both horizontal turbulent viscosity ν_h and horizontal turbulent resistivity η_h are set by β-plane MHD turbulence. The former is central to the S.-Z. scenario, as discussed earlier. Horizontal resistivity is important to the G.-McI. scenario since, in this model, shearing of poloidal, radiation-zone magnetic fields is presumed to be balanced by resistive dissipation of the toroidal, tachocline field. Up until now, only Ohmic, vertical diffusion of magnetic fields has been considered by Gough & McIntyre. Turbulent horizontal diffusion is also possible and is, very likely, a stronger effect (i.e. $-B_o \sin\theta \partial\bar{\Omega}/\partial\theta \sim (\eta\partial_r^2 + \eta_T \partial_h^2)B_\phi$, with η_T dominant).

The model of β-plane MHD turbulence that we employ is simply 2D MHD with the β effect added to the vorticity equation. Using $\mathbf{B} = \nabla A \times \hat{\mathbf{z}}$ and $\mathbf{V} = \nabla\psi \times \hat{\mathbf{z}}$, the governing equations can be written as:

$$\partial_t \nabla^2\psi + \nabla\psi \times \hat{\mathbf{z}} \cdot \nabla\nabla^2\psi + \beta\partial_x\psi = \nabla A \times \hat{\mathbf{z}} \cdot \nabla\nabla^2 A + \nu\nabla^2\nabla^2\psi + \widetilde{f}, \quad (9.1)$$

$$\partial_t A + \nabla\psi \times \hat{\mathbf{z}} \cdot \nabla A = \eta\nabla^2 A + \widetilde{f_a}. \quad (9.2)$$

In this model the magnetic flux function is of the form

$$A = B_0 y + \widetilde{A}, \quad (9.3)$$

so

$$\mathbf{B} = B_0\widehat{\mathbf{x}} + \widetilde{\mathbf{B}}, \tag{9.4}$$

As is the case in β-plane models, β corresponds to the horizontal gradient of the Coriolis parameter, i.e. $\beta = (2\Omega/r_0)\cos\theta_0$, where r_0 is the radius of the shell, Ω is the rotation rate and θ_0 is the latitude at which the β-plane is tangent to the spherical surface.

In this work, $\widehat{\mathbf{x}}$ corresponds to the azimuthal direction, and is the direction of the mean toroidal field; $\widehat{\mathbf{y}}$ corresponds to the polar direction and $\widehat{\mathbf{z}}$ to the radial direction, in which the system is stably stratified. Thus, Rossby waves propagate in the $-\widehat{\mathbf{x}}$ direction, i.e. westward, along $\mathbf{B_0}$. For simplicity we take B_0 to be uniform. Note that in the unforced ($\widetilde{f} \to 0, \widetilde{f}_a \to 0$), inviscid, ideal limit (i.e. ν, $\eta \to 0$), β-plane MHD conserves:

(i) total energy, $E = \langle (\nabla\psi)^2 + (\nabla A)^2 \rangle/2$;
(ii) total A-squared, $H = \langle A^2 \rangle$;
(iii) total cross helicity, $H_c = \langle \nabla\psi \cdot \nabla A \rangle$.

Of course, inclusion of the Lorentz force breaks enstrophy conservation, so two dimensional MHD dynamics is quite different from that of two-dimensional hydrodynamics.

It is interesting to note that even a straightforward linearization of equations (9.1, 9.2) reveals certain fundamental trends in the system. Assuming plane wave solutions and neglecting forcing and dissipation gives the dispersion relation

$$\omega^2 + \omega_{R_\mathbf{k}}\omega - \omega_{A_\mathbf{k}}^2 = 0, \tag{9.5}$$

where

$$\omega_{R_\mathbf{k}} = -\frac{\beta k_x}{k^2}, \tag{9.6}$$

is the Rossby wave frequency, and

$$\omega_{A_\mathbf{k}} = k_x V_{A0}, \tag{9.7}$$

is the Alfvén wave frequency. In β-plane MHD, these two wave branches are coupled. Hence, we note that for $k^2 < \beta/V_{A0}$ (i.e. $\omega_{A\mathbf{k}} < \omega_{R_\mathbf{k}}$), the Rossby wave character is dominant, while for $k^2 > \beta/V_{A0}$ (i.e. $\omega_{R_\mathbf{k}} < \omega_{A_\mathbf{k}}$) the Alfvénic character dominates. Thus, the wavenumber $k_{LR} = (\beta/V_{A0})^{1/2}$ defines a scale which demarks the boundary between Alfvénic and Rossby dominated ranges. For $k < k_{LR}$, the wave spectrum may be thought of as a gas or ensemble of strongly dispersive Rossby waves, while for $k > k_{LR}$, the waves are Alfvén waves. It is well known that even in two dimensions, Alfvén wave turbulence supports a *forward*, rather than inverse cascade, so 'negative viscosity phenomena', such as zonal flow formation, must occur on scales $\ell > \ell_{LR}$, and are thus likely driven by Rossby wave interaction, rather than by turbulence. Note that ℓ_{LR} constitutes a scale which is somewhat reminiscent of the Rhines scale, familiar from discussions of geostrophic turbulence, and so is called the 'linearized Rhines scale' ℓ_{LR} here.

At this point, the critical reader is no doubt motivated to ask: *Since B_0 flips every 11 years while tachocline formation proceeds over 10^6 years, why doesn't B_0 simply 'average out' on dynamically interesting time scales, allowing us to ignore B_0 in consideration of the formation of the tachocline?* This sentiment was more eloquently espoused in the 1998 paper by Gough & McIntyre, who suggested that:

> 'Any field from a putative dynamo in the convection zone could be "dredged" into the tachocline by the meridional flow and thereby influence the dynamics, but it seems unlikely that the rapidly oscillating field associated with the solar cycle would contribute significantly to the dynamics in the radiative zone, particularly in view of the 10^6 year tachocline ventilation time τ_v.'

Here we argue that while this time scale separation may justify ignoring the effects of B_0, *it does not justify the neglect of MHD effects*! The reason is simple - *in high Rm 2D MHD turbulence, $\langle \widetilde{B}^2 \rangle \gg \langle B \rangle^2$, so mean square magnetic fluctuation levels in the system are large, even if the mean field is weak.* This is a direct consequence of the Zeldovich theorem for 2D MHD, which is directly applicable to β-plane MHD (Zeldovich 1957; Diamond *et al.* 2005a; Tobias 2005). Below, we discuss the Zeldovich theorem and its implications.

It is now well known that in 2D high *Rm* MHD turbulence, the magnetic fluctuation intensity usually exceeds the mean field intensity by a large factor. This is a consequence of the stretching of magnetic fields by turbulence, and is encapsulated by the Zeldovich theorem, which follows from the conservation (up to resistive diffusion) of magnetic potential along fluid trajectories in 2D incompressible flow. Here we generalize the Zeldovich theorem to account for direct forcing of the magnetic potential. Note that the presence or absence of β has no explicit impact upon the $\langle A^2 \rangle$ budget. Though we usually do not think of the magnetic field as being stirred, such forcing is quite relevant to tachocline physics, since overshooting plumes naturally entrain convection zone magnetic fields while they plunge into the tachocline from above. During the course of this discussion, we also address and clarify some basic aspects of the Zeldovich theorem.

In 2D incompressible β-plane MHD, the magnetic potential fluctuation satisfies

$$\frac{\partial \widetilde{A}}{\partial t} + \nabla\psi \times \hat{\mathbf{z}} \cdot \nabla \widetilde{A} = -\widetilde{V}_y \langle B \rangle + \eta \nabla^2 \widetilde{A} + \widetilde{f}_a, \tag{9.8}$$

where we have assumed $\langle A \rangle = \langle A(y) \rangle$. Here $\langle B \rangle$ is the mean field and $\widetilde{V}_y = -\partial_x \psi$. Multiplying by $\widetilde{A}$, taking the fluctuation correlation length to be smaller than the scale of $\langle B \rangle$ variation, and integrating over space (denoted by $\langle\ \rangle$) we find

$$\frac{\partial \langle \widetilde{A}^2 \rangle}{\partial t} + \langle \nabla \cdot \mathbf{V} \widetilde{A}^2 \rangle = 2\Big[- \langle \widetilde{V}_y \widetilde{A} \rangle \langle B \rangle + \eta \langle \widetilde{A} \nabla^2 \widetilde{A} \rangle + \langle \widetilde{A} \widetilde{f}_a \rangle \Big]. \tag{9.9}$$

For stationary, homogeneous systems with periodic boundary conditions and no outflow (i.e. keep in mind that a spherical surface constitutes a closed system!), the LHS of equation (9.9) vanishes. Integrating once by parts on the RHS then gives the relation:

$$\langle \widetilde{B}^2 \rangle = \frac{-\langle \widetilde{V}_y \widetilde{A} \rangle \langle B \rangle + \langle \widetilde{A} \widetilde{f}_a \rangle}{\eta}. \tag{9.10}$$

Note that either inhomogeneity or an in/out flow of magnetic potential can significantly alter the balance expressed by equation (9.10). Finally, writing $\langle \widetilde{V}_y \widetilde{A} \rangle$ in the form of a Fick's law (i.e. $\langle \widetilde{V}_y \widetilde{A} \rangle = -\eta_T \partial \langle A \rangle / \partial y = -\eta_T \langle B \rangle$, where η_T is a turbulent resistivity) and noting $d\widetilde{A}/dt = \widetilde{f}_a$ on inertial scales yields

$$\langle \widetilde{B}^2 \rangle = \frac{\eta_T \langle B \rangle^2 + \langle \widetilde{f}_a^2 \rangle \tau_a}{\eta}. \tag{9.11}$$

Here τ_a is the auto-correlation time of the (random) magnetic stirring force $\widetilde{f}_a$.

Equation (9.11) extends the usual Zeldovich theorem balance ($\langle \widetilde{B}^2 \rangle = (\eta_T/\eta) \langle B \rangle^2$) to include the additional effect of random stirring of A. Note that writing $\langle B \rangle^2 = (\partial \langle A \rangle / \partial y)^2$ in equation (9.11) suggests that the total magnetic fluctuation intensity is fed by both:

(i) turbulent mixing of gradients in $\langle A \rangle$ by ambient MHD turbulence, as parameterized by $\eta_T (\partial \langle A \rangle / \partial y)^2$;
(ii) external stirring by overshoot, as parameterized by $\langle \widetilde{f}_a^2 \rangle \tau_a$.

These two stochastic processes are independent, so their contributions to $\langle \widetilde{B}^2 \rangle$ are *additive*. As $\eta_T \gg \eta$, equation (9.11) confirms that $\langle \widetilde{B}^2 \rangle \gg \langle B \rangle^2$, even in the absence of direct stirring of $\widetilde{A}$. Strictly speaking, the Zeldovich theorem balance may be written as

$$\frac{\langle \widetilde{B}^2 \rangle}{\langle B \rangle^2} = N_{u,m} + \langle \widetilde{f}_a^2 \rangle \frac{\tau_a}{\eta}, \tag{9.12}$$

where $N_{u,m}$ is the 'magnetic Nusselt number', η_T/η, and τ_a is the forcing correlation time. In practice, for 2D MHD, $N_{u,m}$ exhibits a strong scaling with magnetic Reynolds number Rm, consistent with both numerical calculations and theoretical expectations (Cattaneo & Vainshtein 1991; Diamond *et al.* 2005a). Note that while η_T is quenched, relative to kinematic based expectations, it still greatly exceeds the collisional resistivity η and so η_T quenching is indeed compatible with $\langle \widetilde{B} \rangle^2 \gg B_0^2$.

Frequently, $N_{u,m} \sim Rm$, though the universality of this putative scaling requires further study and documentation. However, it seems indisputable that in the turbulent tachocline, the β-plane MHD turbulence has $Rm \gg 1$ and so $\langle \widetilde{B}^2 \rangle \gg \langle B \rangle^2$. Thus, the magnetic fluctuations and turbulence dominate the mean magnetic field, rendering the rapid reversals (on tachocline formation time scales) of the mean field direction a moot point. More succinctly put, while $\langle \mathbf{B} \rangle$ may 'average out' over long time scales on account of frequent reversals, $\langle \widetilde{B}^2 \rangle$ will most certainly persist, albeit in different realizations, and in fact be the dominant repository of magnetic energy. Of course, $\widetilde{\mathbf{B}}$ has no systematic directionality. Hence, one should think of the tachocline as magnetized by layers of thin, quasi-2D stochastic magnetic networks (i.e. see Figure 9.2), which support the propagation of Alfvén waves and so 'elasticize' the tachocline layer.

9.3 The Rhines Scale for MHD Turbulence on a β-Plane

A key element in our mental picture of hydrodynamic turbulence on a β-plane is the Rhines scale, which is of significance because it demarks the boundary between small scale 2D turbulence, comprised of eddies and vortices etc. and larger scale Rossby wave turbulence. The physics of the Rhines scale is explained heuristically below. We then proceed to discuss the modifications of the Rhines scale introduced by coupling to stochastic magnetic fields in 2D MHD on the β-plane. We especially focus on possible Rm dependence of the Rhines scale and on its role in separating the region of forward MHD energy cascade from that of transfer of Rossby wave energy by nonlinear wave interaction.

In brief, the Rhines scale of quasi-geostrophic turbulence (Rhines 1975; Diamond *et al.* 2005b) is based on two facts, which are:

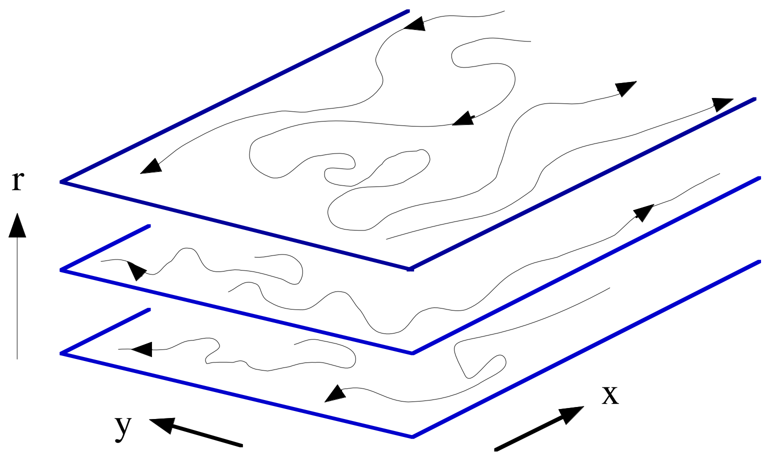

Fig. 9.2. A sketch of turbulent magnetic field structure in the tachocline. The magnetic field is stochastic but organized into thin, quasi-2D layers or shells, on account of the strong, stable stratification. The *mean* magnetic field (not shown here) is primarily in the $\widehat{\mathbf{x}}$ direction.

(i) each $\mathbf{k}$ or scale is characterized by a real frequency $\omega_{\mathbf{k}}$ and a self-decorrelation rate $\Delta\omega_{\mathbf{k}}$. Here $\Delta\omega_{\mathbf{k}}^{-1}$ may be thought of as an effective self-coherence time for the fluctuation with wavevector $\mathbf{k}$. In geostrophic turbulence, the real frequency is approximately the Rossby wave frequency $\omega_{\mathbf{k}} \cong \omega_{R\mathbf{k}} = -\beta k_x/k^2$;

(ii) at long wavelengths, Rossby waves are strongly dispersive, so it is extremely difficult to satisfy the three wave resonance condition for energy transfer unless either:

   (a) one member of the triad has $k_x = 0$ and so is a zonal flow,
   (b) $\Delta\omega_{\mathbf{k}} > \omega_{\mathbf{k}}$, so turbulence interaction effectively smears out wave resonance.

Thus, the scale at which $\omega_{\mathbf{k}} = \Delta\omega_{\mathbf{k}}$ naturally forms a boundary or dividing line between ranges of scales in which the nonlinear energy transfer is controlled by turbulent inverse cascade and resonant wave-wave interaction. This scale is referred to as the *Rhines scale*. Since for (strongly dispersive) Rossby waves, resonant wave interaction occurs only via scattering off an azimuthally symmetric zonal flow mode, the Rhines scale also sets the characteristic width of zonal flows. On dimensional grounds, the Rhines scale is usually estimated by taking $\Delta\omega_{\mathbf{k}} \sim k\widetilde{V}$, so

$$\omega_{\mathbf{k}} \approx k\widetilde{V}, \tag{9.13}$$

implies that the Rhines wave number k_R is given by

$$k_R^2 = \frac{\beta}{\widetilde{V}}, \tag{9.14}$$

and so the Rhines scale $\ell_R \sim (\widetilde{V}/\beta)^{1/2}$. Here $\widetilde{V}$ is a 'typical' eddy velocity, so ℓ_R exhibits some sensitivity to the structure of the spectrum. Moreover, concerns about Galilean invariance have motivated reconsideration of the definition of ℓ_R in terms of the local eddy strain rate (Vallis & Maltrud 1993). The resulting departures from the simple result of equation (9.14) are, however, quite small (Nozawa & Yoden 1997). Thus, the Rhines scale ℓ_R is well established as a useful concept in, and as an element of, descriptions of geostrophic turbulence.

In 2D β-plane MHD at large Rm, $\langle \widetilde{B}^2 \rangle \geq Rm \langle B \rangle^2$ (i.e. for simplicity we now take $N_{u,m} \sim Rm$) so the decorrelation rate $\Delta\omega_{\mathbf{k}}$ is simply $k\widetilde{V}_A$, where $\widetilde{V}_A^2 = \langle \widetilde{B}^2 \rangle / 4\pi\rho_0 \geq Rm V_{A_0}^2$. Here $\widetilde{V}_A$ may be thought of as an effective Alfvén or elastic velocity for propagation in the network of stochastic small scale fields. Such stochastic fields are the principal agents of decorrelation here. Note that since the magnetic field allows large scales to damp small scales by Alfvénic coupling, concerns pertaining to Galilean invariance of the turbulence theory are moot in MHD (Moffatt 1978). Then, since $\widetilde{V}_A \gg V_{A_0}$, the effective boundary between turbulence and Rossby wave ranges in β-plane MHD is given by

$$\frac{\beta k_x}{k^2} \cong k\widetilde{V}_A, \tag{9.15}$$

so the MHD Rhines wave number k_{RM} is given by

$$k_{RM}^2 \cong \frac{\beta}{\widetilde{V}_A}, \tag{9.16}$$

and the effective Rhines scale for β-plane MHD is just $\ell_{RM} \sim (\widetilde{V}_A/\beta)^{1/2}$. As we will see, ℓ_{RM} is an important scale for the dynamics of β-plane MHD turbulence.

Several comments are appropriate here. First, note that ℓ_{RM} is similar to ℓ_{LR} from §9.2, the difference being that $\ell_{RM} \sim (\widetilde{V}_A/\beta)^{1/2}$ while $\ell_{LR} \sim (V_{A_0}/\beta)^{1/2}$, so $\ell_{RM}/\ell_{LR} \gtrsim Rm^{1/4}$. This once again reminds us that coupling to the stochastic small scale magnetic field $\widetilde{\mathbf{B}}$ is much stronger than the coupling to the mean field $\langle \mathbf{B} \rangle$, so the dynamically dominant Alfvén wave propagation is along the stochastic field. This appears in the theory as a decorrelation rate, rather than as a wave frequency, on account of the stochasticity of $\widetilde{\mathbf{B}}$. Second, note that ℓ_{RM} is manifestly Rm dependent, and $\sim Rm^{1/4}$ for the usual scaling of $\eta_T/\eta \sim Rm$. Thus, the range of MHD

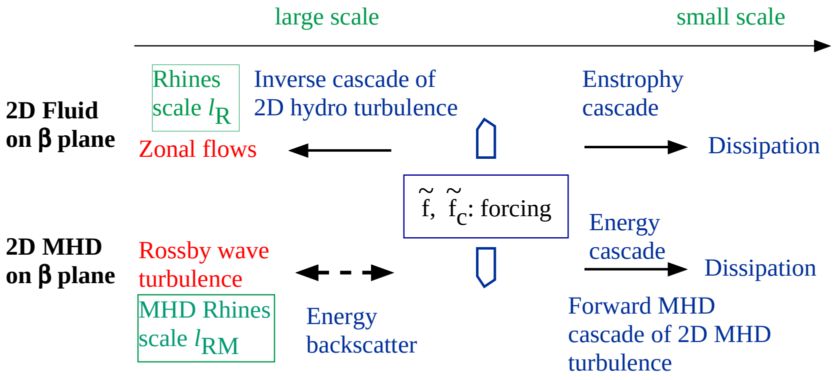

Fig. 9.3. A cartoon contrasting the energy flow in 2D hydrodynamic turbulence on a β-plane with that for 2D MHD turbulence on a β-plane. A large scale stochastic magnetic field lies in the plane.

turbulence (i.e. all scales ℓ such that $\ell_d < \ell < \ell_{RM}$) *increases* with Rm. Third, it is *very important* to keep in mind that the presence of magnetic fields breaks enstrophy conservation, so that MHD turbulence cascades to small scales, even in 2D! In the case where the characteristic forcing scale $\ell_f < \ell_{RM}$, energy will forward cascade to small scale dissipation, while (non-scale-invariant) stochastic backscatter will gradually fill in $\ell_f < \ell < \ell_{RM}$. For $\ell_f > \ell_{RM}$, energy will be transferred forward, though wave interactions will play a role. See Figures 9.3–9.5 for pertinent diagrams. Thus, any energy reaching ℓ_{RM} will eventually be coupled to small scale dissipation. No inverse cascade of energy occurs. Thus, unlike the corresponding case in geostrophic turbulence, there is no reason to expect zonal flow formation at ℓ_{RM}, as *energy does not inverse cascade toward ℓ_{RM} in turbulent β-plane MHD*. Here, the effective Rhines scale ℓ_{RM} merely *separates* the range of forward cascading MHD turbulence from the range dominated by wave interaction.

To summarize this discussion and to understand and gain some perspective on the role of the MHD Rhines scale ℓ_{RM} in β-plane MHD, it is useful to systematically compare and contrast the physics of the Rhines scale ℓ_R familiar from neutral quasi-geostrophic turbulence with that of the MHD Rhines scale ℓ_{RM} for β-plane MHD. Table 9.2 summarizes this comparison. In the case of a neutral geostrophic fluid, the Rhines scale $\ell_R = (\widetilde{V}/\beta)^{1/2}$

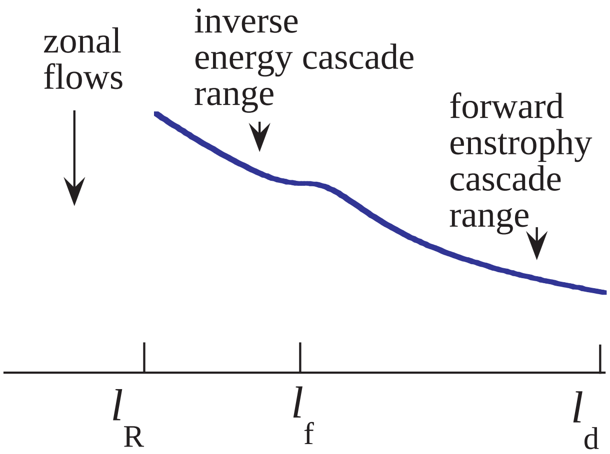

Fig. 9.4. Cartoon of the energy spectrum for geostrophic turbulence. Note that ℓ decreases to the right.

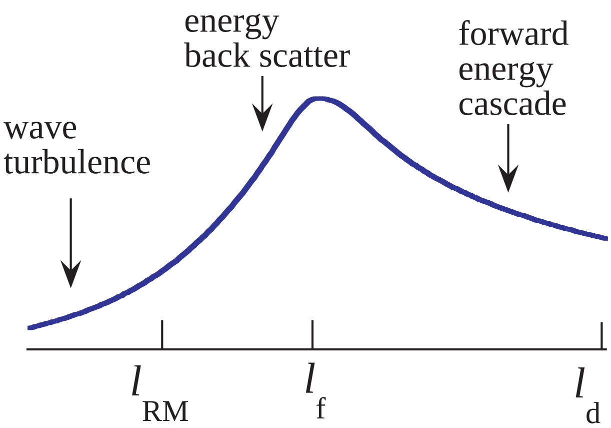

Fig. 9.5. Cartoon of the energy spectrum for β-plane MHD turbulence. Note that ℓ decreases to the right

separates eddy interaction dominated ($\ell < \ell_R$) and wave-zonal flow interaction dominated ranges ($\ell > \ell_R$). For β-plane MHD, the MHD Rhines scale $\ell_{RM} = (\widetilde{V}_A/\beta)^{1/2}$, where $\widetilde{V}_A^2 = \langle \widetilde{B}^2 \rangle / 4\pi\rho_0$, separates an MHD turbulence range ($\ell < \ell_{RM}$) composed of eddies and Alfvén waves from a range with $\ell > \ell_{RM}$, which is dominated by Rossby waves, with some zonal flows and fields present too. In the case of a geostrophic fluid, energy *inverse cascades* toward ℓ_R from all smaller scales, i.e. $\ell < \ell_R$. The strong dispersion of Rossby waves then forces further interaction to proceed primarily via wave-

Table 9.2. *Comparison of Rhines Scales for HD and MHD β-plane Turbulence*

	β-Plane HD	*β-Plane MHD*
Rhines Scale	$\ell_R = (\widetilde{V}/\beta)^{1/2}$	$\ell_{RM} = (\widetilde{V}_A/\beta)^{1/2}$ $\widetilde{V}_A^2 = \langle \widetilde{B}^2 \rangle / 4\pi\rho_0$
Fluctuation Constituents	$\ell < \ell_R$ - eddies $\ell > \ell_R$ - Rossby waves, zonal flows	$\ell < \ell_{RM}$ - Alfvén waves, eddies $\ell > \ell_{RM}$ - Rossby waves, zonal flows, fields
Spectral Energy Flow	$\ell < \ell_R \rightarrow$ INVERSE cascade toward ℓ_R $\ell > \ell_R \rightarrow$ transfer by wave - zonal flow interaction	$\ell < \ell_{RM} \rightarrow$ FORWARD MHD cascade $\ell > \ell_{RM} \rightarrow$ transfer by wave-wave, wave-zonal structure interaction
Structure	strong zonal flows on $\ell \sim \ell_R$ fed by inverse cascade from $\ell < \ell_R$	weak zonal flows and fields on scales $\ell > \ell_{RM}$ fed by wave interaction from scales $\ell > \ell_{RM}$

zonal flow scattering, thus generating zonal flows of characteristic scale ℓ_R. In this case, *nearly all* energy generated on scales with $\ell < \ell_R$ is ultimately fed into large scale zonal flows. Thus, it is no surprise that zonal flows are a prominent feature of such systems. In the case of β-plane MHD, energy in the turbulent range on $\ell < \ell_{RM}$ *forward cascades*, toward small scale dissipation. Energy contained on scales $\ell > \ell_{RM}$ participates in wave-wave and wave-zonal structure (i.e. flow and field) interaction. Note that in the case of β-plane MHD, most of the energy generated on scales $\ell < \ell_{RM}$ does *not* flow to large scales ($\ell > \ell_{RM}$). Such scales are fed only by (non-self-similar) stochastic back-scatter from smaller scales. Thus, we can expect zonal structures to be *significantly* less prominent features in β-plane MHD turbulence then in neutral geostrophic turbulence (Kim, Hahm & Diamond 2001; Naulin *et al.* 2005).

9.4 Spectral Transfer and Turbulent Dissipation in β-Plane MHD

In this section, we examine the dynamics of interactions, spectral transfer and turbulent dissipation (η_h and ν_h) in β-plane MHD. Such a study necessarily builds upon existing understanding of 2D MHD and Rossby wave turbulence, both of which have been extensively studied (Pouquet 1978; Horton 1999). This section should be construed only as an introduction to this large and complex subject, and thus should be viewed as a survey of tachocline-relevant issues in β-plane MHD. Some selected topics are pursued in depth here, but a complete discussion is far beyond the scope of this short article. Here, we shall discuss:

(i) spectral transfer of $\langle \widetilde{A}^2 \rangle_{\mathbf{k}}$ and the turbulent diffusion of magnetic fields;
(ii) the turbulent transport of momentum and turbulent viscosity;
(iii) interactions of an ambient Rossby wave spectrum with a large scale shear flow.

Other aspects of the problem are left for future publications. Throughout this section, we ignore any possible cross-correlation between fluid and magnetic forcing (i.e. we take $\langle \widetilde{f} \widetilde{f}_a \rangle = 0$), so cross helicity may be zeroed *ab initio*. We emphasize, though, that this is only a crude approximation and that the relative coherence of $\widetilde{f}$ and $\widetilde{f}_a$ is an important element of the physics of tachocline turbulence which should be considered carefully (Bracco *et al.* 1998).

Turbulent magnetic dissipation is best addressed by examining the spectral dynamics of $\langle \widetilde{A}^2 \rangle$. The variance of the magnetic potential evolves according to

$$\frac{\partial}{\partial t} \frac{\langle \widetilde{A}^2 \rangle}{2} + \frac{\langle \nabla \psi \times \widehat{\mathbf{z}} \cdot \nabla \widetilde{A}^2 \rangle}{2} = -\eta \langle \widetilde{B}^2 \rangle + \langle B \rangle \langle \widetilde{A} \partial_x \psi \rangle + \langle \widetilde{A} \widetilde{f}_a \rangle, \tag{9.17}$$

or, equivalently, in k space

$$\frac{\partial}{\partial t} \langle \widetilde{A}^2 \rangle_{\mathbf{k}} + T_{\mathbf{k}} = 2 \left[\eta_{T\mathbf{k}} \left(\frac{\partial \langle A \rangle}{\partial y} \right)^2 + \langle \widetilde{f}_a^2 \rangle_{\mathbf{k}} \tau_{a\mathbf{k}} - \eta \langle \widetilde{B}^2 \rangle_{\mathbf{k}} \right], \tag{9.18}$$

where the nonlinear transfer term $T_{\mathbf{k}}$ is:

$$T_{\mathbf{k}} = \langle \nabla \psi \times \widehat{\mathbf{z}} \cdot \nabla \widetilde{A}^2 \rangle_{\mathbf{k}}. \tag{9.19}$$

Note the terms on the RHS of equation (9.18) are precisely those which determine the Zeldovich theorem relation. Hereafter we refer to the RHS of equation (9.18) as $\mathcal{S}_{\mathbf{k}}$, the net *source* for the time evolution of $\langle \widetilde{A}^2 \rangle_{\mathbf{k}}$. The Zeldovich theorem, then, is merely a statement that $\sum_{\mathbf{k}} \mathcal{S}_{\mathbf{k}} = 0$, so that there is a net balance of inflow and outflow of $\langle \widetilde{A}^2 \rangle$.

We now calculate $T_{\mathbf{k}}$ via closure theory (Orszag 1970; Yoshizawa, Itoh & Itoh 2003). Before proceeding to grind the crank, some general comments on $\langle \widetilde{A}^2 \rangle_{\mathbf{k}}$ transfer are in order. It is well known that in 2D MHD, magnetic potential isocontours:

(i) tend to be 'chopped up' by the turbulent flow until reaching small scale dissipation, or until Lorentz force back-reaction becomes significant;

(ii) tend to coalesce and aggregate on large scales, on account of the mutual attraction of like-signed current filaments.

The competition between these two processes determines the net effective magnetic dissipation. Since 'chopping up' tends to win if $\langle\widetilde{V}^2\rangle \gg \langle\widetilde{B}^2\rangle$ while coalescence tends to win if $\langle\widetilde{B}^2\rangle \gg \langle\widetilde{V}^2\rangle$, the net turbulent magnetic dissipation tends to scale as

$$\eta_T \sim (\langle\widetilde{V}^2\rangle - \langle\widetilde{V}_A^2\rangle)\tau_c, \tag{9.20}$$

where τ_c is a correlation time. Such a form for η_T has indeed been recovered from the results of renormalized closure theory (Pouquet 1978). Subsequent use of the Zeldovich theorem balance then gives the form of the 'quenched' magnetic diffusivity in 2D MHD:

$$\eta_T \sim \frac{\eta^k}{1 + RmV_{A0}^2/\langle\widetilde{V}^2\rangle}, \tag{9.21}$$

where η^k is the familiar kinematic diffusivity $\eta^k \sim \langle\widetilde{V}^2\rangle\tau_c$ (Pouquet, Frisch & Léorat 1976). In the interesting limit where $RmV_{A0}^2/\langle\widetilde{V}^2\rangle \gg 1$, the corresponding limit of equation (9.21) can be obtained directly from the Zeldovich theorem, with the additional reasonable assumption of approximate equipartition, so that $\eta_T \sim \eta\langle\widetilde{V}^2\rangle/\langle B\rangle^2$.

The main issues to be addressed here are:

(i) just what exactly is τ_c and how is it determined?

(ii) what are the effects of Rossby wave coupling on spectral transfer of $\langle\widetilde{A}^2\rangle$?

Regarding (i), we have previously discussed the physics of τ_c, which is Alfvénic propagation along a fluctuating network of stochastic magnetic fields, so $1/\tau_{c\mathbf{k}} = \Delta\omega_{\mathbf{k}} = k\widetilde{V}_A$. Note that τ_c is itself necessarily slowly time-dependent. To assess the effects of Rossby coupling (i.e. item (ii)), a closure calculation is necessary.

Here, we present an eddy-damped quasi-normal Markovian (EDQNM) closure calculation of $T_{\mathbf{k}}$ for β-plane MHD. The EDQNM closure develops a set of coupled spectra equations from the assumption of weakly non-Gaussian mode amplitude statistics and from physically motivated choices (further details can be found in Orszag 1970; Pouquet 1978).

Writing $T_{\mathbf{k}}$ as

$$\begin{aligned} T_{\mathbf{k}} &= \Big\langle \sum_{\mathbf{k}'} (\mathbf{k}\cdot\mathbf{k}'\times\widehat{\mathbf{z}})(\widetilde{A}_{-\mathbf{k}}\widetilde{\psi}_{-\mathbf{k}'}\widetilde{A}^{(2)}_{\mathbf{k}+\mathbf{k}'} + \widetilde{A}_{-\mathbf{k}}\widetilde{A}_{-\mathbf{k}'}\widetilde{\psi}^{(2)}_{\mathbf{k}+\mathbf{k}'}) \\ &- \sum_{\substack{\mathbf{p},\mathbf{q} \\ \mathbf{p}+\mathbf{q}=\mathbf{k}}} (\mathbf{p}\cdot\mathbf{q}\times\widehat{\mathbf{z}})\widetilde{\psi}_{-\mathbf{p}}\widetilde{A}_{-\mathbf{q}}\widehat{A}^{(2)}_{\mathbf{p}+\mathbf{q}}\Big\rangle, \end{aligned} \tag{9.22}$$

we seek to calculate $\widetilde{A}^{(2)}_{\mathbf{k}+\mathbf{k}'}$ and $\widetilde{\psi}^{(2)}_{\mathbf{k}+\mathbf{k}'}$ such that $T_{\mathbf{k}}$ is independent of fluctuation phase. The calculation is simplified by taking $\langle\mathbf{V}\cdot\mathbf{B}\rangle = 0$ and ignoring $\langle\mathbf{B}\rangle$ relative to $\widetilde{B}_{rms}$, which amounts to neglecting linear Alfvén wave propagation in comparison to stochastic Alfvénic decorrelation (i.e. since $kV_{A0} < \Delta\omega_{\mathbf{k}}$). Thus, $\widetilde{A}^{(2)}_{\mathbf{k}+\mathbf{k}'}$ and $\widetilde{\psi}^{(2)}_{\mathbf{k}+\mathbf{k}'}$ are written as:

$$\widetilde{A}^{(2)}_{\mathbf{k}+\mathbf{k}'} = \int_{-\infty}^{t} dt' e^{-\Delta\omega_{\mathbf{k}+\mathbf{k}'}(t-t')}(\mathbf{k}\cdot\mathbf{k}'\times\widehat{\mathbf{z}})\widetilde{\psi}_{\mathbf{k}'}(t')\widetilde{A}_{\mathbf{k}}(t'), \tag{9.23}$$

$$\widetilde{\psi}^{(2)}_{\mathbf{k}+\mathbf{k}'} = \int_{-\infty}^{t} dt' e^{-(i\omega_{\mathbf{k}+\mathbf{k}'}+\Delta\omega_{\mathbf{k}+\mathbf{k}'})(t-t')}(\mathbf{k}\cdot\mathbf{k}'\times\widehat{\mathbf{z}})\frac{(k^2-k'^2)}{k''^2}\widetilde{A}_{\mathbf{k}'}(t')\widetilde{A}_{\mathbf{k}}(t'). \tag{9.24}$$

$\widetilde{A}^{(2)}_{\mathbf{p}+\mathbf{q}}$ is identical to $\widetilde{A}^{(2)}_{\mathbf{k}+\mathbf{k}'}$, up to a re-labeling. The two-point time correlators for $\widetilde{\psi}$ and $\widetilde{A}$ (here written for some generic field F) are taken to have the approximate structure:

$$\langle F^*_{\mathbf{k}}(t)F_{\mathbf{k}}(t')\rangle = |F_{\mathbf{k}}(t)|^2 e^{-(i\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}})(t-t')}. \tag{9.25}$$

Here, rapid decay on the $(\Delta\omega_{\mathbf{k}})^{-1}$ time scale accounts for decorrelation of resonant triads due to nonlinear scrambling, while the slower envelope behavior (i.e. $|F_{\mathbf{k}}(t)|^2$) accounts for evolution of the spectrum in time. Given all this, it follows that the renormalized $\langle\widetilde{A}^2\rangle_{\mathbf{k}}$ transfer rate $T_{\mathbf{k}}$ is given by:

$$\begin{aligned} T_{\mathbf{k}} &= \sum_{\mathbf{k}'} (\mathbf{k}\cdot\mathbf{k}'\times\widehat{\mathbf{z}})^2\theta^A_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'}|\widetilde{\psi}_{\mathbf{k}'}(t)|^2|\widetilde{A}_{\mathbf{k}}(t)|^2 \\ &+ \sum_{\mathbf{k}'} (\mathbf{k}\cdot\mathbf{k}'\times\widehat{\mathbf{z}})^2\theta^\psi_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'}\left(\frac{k^2-k'^2}{k''^2}\right)|\widetilde{A}_{\mathbf{k}'}(t)|^2|\widetilde{A}_{\mathbf{k}}(t)|^2 \\ &- \sum_{\substack{\mathbf{p},\mathbf{q} \\ \mathbf{p}+\mathbf{q}=\mathbf{k}}} (\mathbf{p}\cdot\mathbf{q}\times\widehat{\mathbf{z}})^2\theta^A_{\mathbf{p},\mathbf{q},\mathbf{k}'}|\widetilde{\psi}_{\mathbf{p}}(t)|^2|\widetilde{A}_{\mathbf{q}}(t)|^2, \end{aligned} \tag{9.26}$$

where $k''^2 = (\mathbf{k}+\mathbf{k}')^2$ and

$$\theta^A_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'} = \Re\left\{\frac{i}{((\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}) + i(\Delta\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}'}+\Delta\omega_{\mathbf{k}+\mathbf{k}'}))}\right\}, \tag{9.27}$$

$$\theta^\psi_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'} = \Re\left\{\frac{i}{((\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}+\mathbf{k}'}) + i(\Delta\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}'}+\Delta\omega_{\mathbf{k}+\mathbf{k}'}))}\right\}. \tag{9.28}$$

Here θ^A represents the triad coherence time for the first and third terms in the expression for $T_{\mathbf{k}}$ while θ^ψ represents that for the second. Equations (9.26), (9.27) and (9.28) then give the full result for the renormalized $\langle \widetilde{A}^2 \rangle_{\mathbf{k}}$ transfer rate, $T_{\mathbf{k}}$.

Several aspects of equations (9.26) – (9.28) merit discussion at this point. First, note that $T_{\mathbf{k}}$ has the usual structure of coherent damping terms (i.e. the first two) competing against incoherent emission (i.e. the third; Kraichnan 1959). The coherent damping terms determine the effective turbulent magnetic dissipation. Thus, we have the turbulent magnetic diffusivity:

$$\eta_T \cong \sum_{\mathbf{k}'} k'^2 [\theta^A_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'} |\widetilde{\psi}_{\mathbf{k}'}(t)|^2 - \theta^\psi_{\mathbf{k},\mathbf{k}',\mathbf{k}+\mathbf{k}'} |\widetilde{A}_{\mathbf{k}'}(t)|^2]. \tag{9.29}$$

Here we have taken $|\mathbf{k}| \ll |\mathbf{k}'|$, (i.e. we consider the dissipation of larger scales than the scale on which the system is forced) so $(k^2 - k'^2)/k''^2 \to -1$. It is not surprising to see that η_T for β-plane MHD is quite similar to its counterpart for 2D MHD, apart from the triad coherence factors θ^A and θ^ψ, which contain the wave frequency contributions. Note that there is a slight difference between the frequency dependencies of θ^A and θ^ψ. In particular, θ^ψ is considerably more sensitive to Rossby wave dispersion, in that $\omega_{\mathbf{k}} + \omega_{\mathbf{k}'} \simeq 0$ is easily satisfied but triad resonance, as in θ^ψ, is not. A detailed quantitative study of the implication of $\theta^A \neq \theta^\psi$ is beyond the scope of this chapter.

Several aspects of equation (9.29) merit further discussion, as well. First, and most important, it is easy to see that for $\Delta\omega > \omega_{\mathbf{k}} + \omega_{\mathbf{k}'}$, $\Delta\omega > \omega_{\mathbf{k}} + \omega_{\mathbf{k}'} - \omega_{\mathbf{k}+\mathbf{k}'}$ (where $\Delta\omega$ refers to the sum of the three model decorrelation rates in θ^A and θ^ψ) equation (9.29) passes smoothly to results previously obtained for 2D MHD with $\beta = 0$ (Diamond *et al.* 2005a). This limit corresponds to length scales $\ell < \ell_{RM}$. Thus, existing results from the theory of turbulent diffusion of magnetic fields in 2D MHD predict that the horizontal turbulent resistivity η_T is (strongly) quenched, in comparison to kinematic turbulence predictions, i.e.

$$\eta_T = \frac{\eta^k}{1 + Rm V_{A0}^2 / \langle \widetilde{V}^2 \rangle}. \tag{9.30}$$

For $Rm V_{A0}^2/\langle \widetilde{V}^2 \rangle > 1$, η_T is well approximated by $\eta_T \approx \eta \langle \widetilde{B}^2 \rangle / \langle B \rangle^2$. For $\langle \widetilde{V}^2 \rangle \sim \langle \widetilde{V}_A^2 \rangle$, these are equivalent to $\eta_T \approx \eta \langle \widetilde{V}^2 \rangle / V_{A0}^2$. *The reader should take note, however, that while η_h is quenched relative to the standard kinematic estimates, it still greatly exceeds the collisional magnetic diffusivity η, since $\langle \widetilde{B}^2 \rangle \gg \langle B \rangle^2$ etc.* Thus, *turbulent horizontal diffusion of magnetic fields* may still be a significant mechanism for dissipating magnetic energy in the solar tachocline. We will discuss this issue further in the concluding section.

Having addressed the question of turbulent resistivity, we now consider the physics of turbulent viscosity and turbulent momentum transport in β-plane MHD. Recall that in the S.-Z. scenario of tachocline formation, the tachocline location in the solar core is determined by the balance of burrowing with horizontal transport (in latitude) of momentum. In β-plane MHD, the mean azimuthal flow evolves according to

$$\frac{\partial}{\partial t}\langle V_x\rangle = \frac{-\partial}{\partial y}\left\{\langle \widetilde{V}_y\widetilde{V}_x\rangle - \frac{\langle \widetilde{B}_y\widetilde{B}_x\rangle}{4\pi\rho_0}\right\}, \tag{9.31}$$

where we have ignored molecular viscosity and the momentum source related to burrowing. Also, here ρ_0 is taken as a constant as we consider incompressible 2D dynamics. Note that in MHD, the net flux transport is determined by the *difference* between fluid and magnetic stresses. Thus, in a perfectly Alfvénized state the total momentum flux *vanishes*. Of course, the traditionally invoked turbulent horizontal viscosity ν_h is based upon a 'mixing length' approximation to the fluid stress $\langle \widetilde{V}_y\widetilde{V}_x\rangle$, which is constructed by asserting that fluctuations in $\widetilde{V}_y$ occur via mixing of $\langle V_x\rangle$, i.e.

$$\begin{aligned}\widetilde{V}_y &= \langle V_x(y-\ell)\rangle - \langle V_x(y)\rangle \\ &\cong -\ell\frac{\partial\langle V_x\rangle}{\partial y},\end{aligned} \tag{9.32}$$

so

$$\begin{aligned}\langle \widetilde{V}_y\widetilde{V}_x\rangle &= -(\widetilde{V}_x\ell)\frac{\partial\langle V_x\rangle}{\partial y} \\ &\equiv -\nu_h\frac{\partial\langle V_x\rangle}{\partial y}.\end{aligned} \tag{9.33}$$

Here ℓ is the mixing length. Note that the main novel feature in the case of MHD is the competition between the two stresses. Thus, we focus our attention on this competition. It is useful to split the integration over scales (implicit in the averages in equation (1.31)) into ranges of $k_<$ and $k_>$, where the $k_<$-range includes all $\mathbf{k}$ such that $|\mathbf{k}| < k_{RM} = 2\pi/\ell_{RM}$ and the $k_>$-range includes all $\mathbf{k}$ such that $|\mathbf{k}| > k_{RM}$. Thus, the $k_>$-range corresponds to the range of *forward* cascading of energy in 2D MHD turbulence for which wave interactions are subdominant, while the $k_<$-range is the range where nonlinear transfer etc. are controlled by *Rossby-Alfvén wave interaction* (see Figure 9.5). Denoting the total momentum flux by Γ_v, we thus can write:

$$\frac{\partial\langle V_x\rangle}{\partial t} = -\frac{\partial}{\partial y}\Gamma_v$$

$$= \quad -\frac{\partial}{\partial y}\Big(\sum_{k_<}\Gamma_{v\mathbf{k}} + \sum_{k_>}\Gamma_{v\mathbf{k}}\Big), \tag{9.34}$$

where

$$\Gamma_{v_\mathbf{k}} = -k_x k_y(|\psi_\mathbf{k}|^2 - |A_\mathbf{k}|^2) \tag{9.35}$$

and the y-dependence of Γ_v is, by definition, 'slow', as it corresponds to variation on scales larger than those typical of the turbulence.

We now discuss the contributions to the momentum flux coming from the $k_>$ and $k_<$-ranges. The $k_>$-range exhibits two-dimensional MHD-like turbulence dynamics. One of the most robust features of 2D MHD is the trend toward approximate equipartition between hydrodynamic and magnetic energy on inertial range scales, i.e. $|\widetilde{\mathbf{V}}_\mathbf{k}|^2 \simeq |\widetilde{\mathbf{B}}_\mathbf{k}|^2$ (Pouquet 1978). This is yet another manifestation of 'Alfvénization', a ubiquitous feature of MHD turbulence. Apart from a small 'residual energy', significant departure from equipartition is due only to those effects which force an *imbalance* between fluid and field, such as deviation of the magnetic Prandtl number from unity (i.e. $P_m \neq 1$), differences between $\langle\widetilde{f}^2\rangle_\mathbf{k}, \langle\widetilde{f}_a^2\rangle_\mathbf{k}$, etc. In the tachocline $P_m \leq 1$ but not drastically so, and $\langle\widetilde{f}^2\rangle_\mathbf{k}, \langle\widetilde{f}_a^2\rangle_\mathbf{k}$ must have finite correlation, as both are due to convective overshoot and its consequent entrainment of convection zone magnetic fields. Thus, it seems eminently reasonable to expect significant competition and cancelation between fluid and magnetic stresses in the $k_>$-range, resulting in a substantial shortfall in turbulent momentum transport, relative to expectations. This tendency toward cancelation is a trivial consequence of the fluid and magnetic stresses tending toward equality and entering Γ_v with opposite sign. Thus, we can write $\Gamma_{v_>}$, the momentum flux due to fluctuations in the $k_>$-range, as

$$\Gamma_{v_>} = \sum_{k_>} -k_x k_y r_\mathbf{k}|\psi_\mathbf{k}|^2 \simeq r\langle\widetilde{V}_y\widetilde{V}_x\rangle, \tag{9.36}$$

where $r_\mathbf{k} \ll 1$, and is the 'residual' factor, dependent upon the quantities causing imbalances, so that,

$$r_\mathbf{k} = r_\mathbf{k}(P_m, \langle\widetilde{f}^2\rangle, \langle\widetilde{f}_a^2\rangle, ...) \ll 1, \tag{9.37}$$

means that the mean flow is effectively 'laminarized', and the turbulent viscosity (due to $|\mathbf{k}| > k_{RM}$) is effectively 'quenched' (note that the sign of $r_\mathbf{k}$ may vary with $\mathbf{k}$). The reduction of momentum transport due to Alfvénization of turbulence is well known in the context of the theory of 'ω-quenching' (Craddock & Diamond 1991; Küker, Rüdiger & Kitchatinov 1993; Kim & Dubrulle 2001; Kim, Hahm & Diamond 2001) and in relation to the reduction of the rate of zonal flow generation as drift-Alfvén turbulence becomes more Alfvénic in character (Naulin *et al.* 2005). The upshot of this trend is that the contribution of the $k_>$-range to ν_h will likely be feeble in the absence of some mechanism which feeds the imbalance between fluid and magnetic energies, such as the magneto-rotational instability.

On scales $|\mathbf{k}| < k_{RM}$ (i.e. in the $k_<$-range), the fluctuation characteristics are predominantly those of Rossby waves, with quite modest magnetic perturbations. The wave turbulence nature of the $k_<$-range fluctuations precludes direct application of the 'conventional wisdom' of strong hydrodynamic turbulence in 2D. Here, we explore the interaction of an ambient Rossby wave spectrum with a weak, large scale 'test' shear spectrum as a means to ascertain the nature of the effective viscosity of a Rossby wave gas. Specifically, should the waves *gain* energy from the shear, the effective viscosity is *positive*, while if the waves *lose* energy to the shear, the effective viscosity is *negative*. Therefore, we proceed by examining the *modulational stability* of an ambient spectrum or gas of Rossby waves to a large scale shear flow perturbation (Diamond *et al.* 2005b).

Noting that large scale magnetic fluctuations are weak, that the Rossby wave energy density $\mathcal{E}_\mathbf{k} = k^2|\psi_\mathbf{k}|^2$, and considering a weak, large scale test flow δV_x, we have from equation (9.31),

$$\frac{\partial}{\partial t}\delta V_x = \frac{\partial}{\partial y}\sum_{\mathbf{k}_<}\frac{k_x k_y}{k^2}\widetilde{\mathcal{E}}_\mathbf{k}, \tag{9.38}$$

where $\widetilde{\mathcal{E}}_\mathbf{k}$ indicates the *modulation* in the wave energy induced by δV_x. Since the Rossby wave population density N is simply the enstrophy density (Dubrulle & Nazarenko 1997), we can re-write equation (9.38) as

$$\frac{\partial}{\partial t}\delta V_x = \frac{\partial}{\partial y}\sum_{\mathbf{k}_<}\frac{k_x k_y}{k^4}\widetilde{N}_\mathbf{k}, \tag{9.39}$$

where

$$\frac{\partial \widetilde{N}}{\partial t} + \mathbf{v}_g\cdot\nabla\widetilde{N} + \delta\omega_\mathbf{k}\widetilde{N} = \frac{\partial(k_x\delta V_x)}{\partial y}\frac{\partial\langle N\rangle}{\partial k_y} \tag{9.40}$$

is the linearized wave kinetic equation for $\widetilde{N}$. The wave kinetic equation simply states that wave population density is an adiabatic invariant for slowly varying, large scale shear flows and is conserved along rays, up to scrambling. The population perturbation varies adiabatically with the large scale flow perturbation. Here $\delta\omega_\mathbf{k}$ accounts for the finite lifetime of a wave packet induced by nonlinear scrambling, $\mathbf{v}_g$ is the packet group velocity, and $\langle N\rangle$ is the mean Rossby wave enstrophy spectrum. Note that equation (9.40) de-

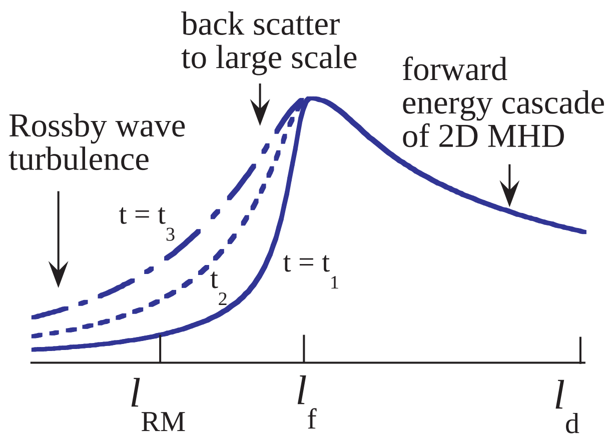

Fig. 9.6. A cartoon showing the development in time of the large scale portion of the energy spectrum for β-plane MHD turbulence. Note that since the Rossby wave scales satisfy $\ell_R < \ell_{RM} < \ell_f$ and since the MHD energy cascade is *forward* from ℓ_f, the spectral slope is non-negative in the Rossby wave dominated range of scales. Note that scale decreases towards the right.

termines the modulation in N induced by the weak 'test' shear flow. Writing

$$\delta V_x = \sum_{\mathbf{q},\Omega} \widetilde{V}_{\mathbf{q},\Omega} e^{i(\mathbf{q}\cdot\mathbf{x}-\Omega t)}, \tag{9.41}$$

a straightforward calculation gives

$$\Im\{\Omega_{\mathbf{q}}\} = -q_y^2 \sum_{\mathbf{k}} \left(\frac{k_x^2 k_y}{k^4}\right) \frac{\delta\omega_{\mathbf{k}}}{(\Omega - \mathbf{q}.\mathbf{v}_{gr})^2 + \delta\omega_{\mathbf{k}}^2} \frac{\partial\langle N\rangle}{\partial k_y}, \tag{9.42}$$

as the rate of growth of the test shear. Thus, for $\partial\langle N\rangle/\partial k_y < 0$ the shear is *amplified*, hence indicating a *negative viscosity*. Note this is the case for the forward enstrophy cascade range, for which $\partial\langle N\rangle/\partial k < 0$. However, if $\partial\langle N\rangle/\partial k_y > 0$, the *shear gives energy to the turbulence* (i.e. *the shear is damped*), so the effective viscosity is *positive*. In the likely event that the forcing spectrum $\langle\widetilde{f}^2\rangle_{\mathbf{k}}$ peaks on small scales, i.e. on $\ell < \ell_{RM}$, the $k_<$-range is energized by the *backscatter* of energy toward large scale Rossby waves, as shown in Figure 9.6. The enstrophy is thus necessarily *increasing* with $\mathbf{k}$ in the $k_<$-range, so $\partial\langle N\rangle/\partial k_y > 0$ is possible there and the effective viscosity would therefore be *positive*. The value of ν_h departs considerably from simplistic expectations and is smaller than standard estimates by the factor $(\delta\omega/\mathbf{q}\mathbf{v}_{gr})^2 < 1$.

At this point it seems fair to say that the nature of the effective turbulent viscosity of β-plane MHD turbulence is a subtle question, indeed!

Having divided the set of scales available to excitation into two classes, the $k_<$-range and the $k_>$-range, we have seen that the (directly excited) $k_>$-range contributes little to the turbulent viscosity, on account of the process of Alfvénization and the consequent near-cancelation of fluid and magnetic stresses. Rossby wave turbulence in the $k_<$-range generates a positive viscosity, but one which is small in $(\delta\omega/\mathbf{q}\mathbf{v}_{gr})^2$. Thus, the effective turbulent viscosity is substantially reduced or quenched, in comparison to expectations. It is also amusing to note that our predictions concerning ν_h disagree with those of *both* Spiegel & Zahn and Gough & McIntyre! Recall that S.-Z. assume a *positive* viscosity, linked to tachocline excitation by plumes and to various large scale flow profile driven instabilities, which were subsequently investigated in some detail (Chaboyer & Zahn 1992; Zahn 1992). Building upon the conventional intuition for 2D quasi-geostrophic turbulence, G.-McI. argue that hydrodynamic turbulence will produce a *negative* viscosity, and will drive potential vorticity homogenization (Rhines & Young 1982). G.-McI. then use this argument to further claim that hydrodynamic turbulence cannot sustain a stationary tachocline against spin-down driven 'burrowing', thus bolstering their argument that a magnetic field must exist in the radiative core of the Sun in order to restrict tachocline penetration. We argue, however, that most scales ($\ell \lesssim \ell_{RM}$) contribute *essentially nothing* to turbulent viscosity, since Alfvénization of the β-plane MHD turbulence results in near-cancelation of fluid and magnetic stresses, as in the case of ω-quenching. We do suggest the possibility that energy exchange between the large scale Rossby wave spectrum and the mean flow may persist. This interaction may be (loosely) thought of as a 'viscosity'. However, in contrast to the standard simple mixing models, such wave-flow interaction is quite sensitive to the structure of the Rossby wave spectrum at large ($\ell > \ell_{RM}$) scales. The wave spectrum structure in this range emerges from stochastic backscatter from smaller scales.

9.5 Discussion and Conclusions

This paper has discussed the physics of turbulent dissipation in the solar tachocline. In this paper, we have identified β-plane MHD as the 'minimal model' of tachocline turbulence, and have investigated the mechanisms of energy transfer, turbulent transport and dissipation of mechanical and magnetic energy according to this model. The principal results are:

(i) a key scale, ℓ_{RM}, that demarks the boundary between 2D MHD dynamics and Rossby wave dynamics was identified. This scale is somewhat analogous to the Rhines scale, familiar from hydrodynamic geostrophic turbulence. However, for $\ell < \ell_{RM}$, a forward cascade of energy occurs in β-plane MHD. Moreover, ℓ_{RM} depends upon magnetic Reynolds number Rm;

(ii) turbulence on scales $\ell < \ell_{RM}$ tends to 'Alfvénize', and thus will not substantially mix and transport momentum. These scales do *not* contribute to ν_h. This is a consequence of close competition between fluid and magnetic stresses. Some momentum transport due to the nonlinear interaction of large scale Rossby waves may occur;

(iii) turbulent resistivity is quenched in β-plane MHD, as in 2D MHD. However, even the 'quenched' η_h greatly exceeds the collisional resistivity. Thus, turbulent *horizontal* diffusive dissipation (η_h) of magnetic fields in the tachocline may pose a significant limitation on tachocline penetration.

The implications of these results for tachocline formation models require some discussion. Indeed, the results and ideas presented here may not be a welcome addition to the theory of the solar tachocline! The 'turbulent horizontal viscosity' invoked in the S.-Z. scenario is *problematic*. 'Generic' turbulence on scales $\ell < \ell_{RM}$ will not significantly mix or transport momentum to a large extent. Rossby waves on scales $\ell > \ell_{RM}$ may drive some transport, but this process is *very* sensitive to the structure of the Rossby wave spectrum and depends upon the large-scale, low-energy tail of the spectrum, about which very little is known. Alternatively, some flow profile-driven instability may produce a ν_h, but despite extensive study, the specific mechanism involved has yet to be identified. Also, proponents of this type of viscosity mechanism must explain why such an instability will not simply hover near marginality, producing a state of 'self-organized criticality' rather than steady viscous dissipation (Diamond & Hahm 1995). Work on simple systems has shown that transport and relaxation in a continuum SOC are *not* well modeled by simple diffusion (Hwa & Kardar 1992). Thus, considerable clarification of the dynamics underlying the 'horizontal viscosity' invoked in the S.-Z. scenario is necessary in order to solidify the foundations of that model.

In the case of the G.-McI. scenario, the principal effect of turbulence is to introduce turbulent horizontal diffusion of magnetic fields, so that the balance of shearing of poloidal field B_0 with dissipation of toroidal field B_ψ now becomes

$$-B_0 \sin\theta \frac{\partial \bar{\Omega}}{\partial \theta} = \left(\eta \partial_r^2 + \eta_T \partial_h^2\right) B_\psi, \tag{9.43}$$

where ∂_h refers to a horizontal derivative. Here $\eta_T/\eta \approx \langle \widetilde{B}^2 \rangle / \langle B \rangle^2 \sim Rm$, so even the 'quenched' turbulent resistivity greatly exceeds the collisional value. The 'bottom line' here is that for $\eta_T/\eta \sim Rm > r_0^2/\Delta_T^2$, where r_0 is the tachocline radius and Δ_T its thickness, *turbulent horizontal diffusion of magnetic fields will dissipate magnetic energy faster than radial collisional resistive diffusion does, as assumed in the G.-McI. scenario.* Since $r_0 \sim 0.7R_\odot$ and $\Delta_T \sim 0.03R_\odot$, in practice this means that for $Rm \gtrsim 500$ (certainly satisfied in the solar tachocline!) horizontal turbulent diffusion is *the* dissipation process which limits tachocline burrowing. Here Rm is, of course, the Reynolds number for the horizontal motion, i.e. $Rm = v_h l/\eta$. Hence, the scaling of the tachocline thickness and its dependence on B_0 (predicted by Gough & McIntyre) should be reconsidered in the light of this observation.

This paper poses more questions than it answers. Indeed, it should be viewed only as an introduction to the problem of β-plane MHD turbulence in the tachocline. Several future investigations are strongly suggested. These include, but are not limited to:

(i) completion of a rigorous analysis in the vein begun here, and accompanied by related numerical simulations which test the theory;
(ii) examining the effects of $\langle \widetilde{f}\widetilde{f}_a \rangle \neq 0$ correlations and finite cross helicity on the turbulent dissipation processes (Bracco *et al.* 1998);
(iii) study of a two-layer β-plane MHD model, in which only the upper layer is forced by convective overshoot;
(iv) extension of this study to geostrophic MHD turbulence on a sphere and in a spherical shell. In this regard, we note that the existing large scale numerical calculations of tachocline dynamics are purely hydrodynamic (Miesch 2001, 2003 and Chapter 5);
(v) consideration of large scale flow structure and its coupling to tachocline turbulence dynamics (Gilman & Fox 1997);
(vi) study of the types and physics of coherent magnetic structures formed in tachocline turbulence. Such structures may have the form of magnetic vortices (Kinney, McWilliams & Tajima 1995; Gruzinov, Das & Diamond 2002) or zonal magnetic fields (Gruzinov *et al.* 2002). It has been suggested that magnetic structures formed in the tachocline may leave an 'imprint' on the magnetic fields ultimately observed in the photosphere (Spiegel, private communication);

(vii) consideration of the effects of tachocline turbulence and flow structure on the solar differential rotation (Itoh *et al.* 2005);

(viii) study of the dissipation of magnetic fields by *vertical mixing* in the stably stratified tachocline. Internal wave interaction is a possible agent of such mixing.

Of course, more consideration should also be given to the physics of vertical mixing in the strong, stable stratification environment of the tachocline. In particular, the effects of wave interactions on vertical transport of magnetic fields and on vertical momentum transport (including possible 'negative viscosity effects') merit further investigation. We hope that such studies will help elucidate the structure of the solar tachocline.

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References

Bracco, A., Provenzale, A., Spiegel, E.A. & Yecko, P. (1998). *'Spotted Disks'*, astro-ph/9802298 23 Feb.

Bretherton, F. & Spiegel, E.A. (1968). *Astrophys. J.* **153**, L77.

Cattaneo, F. & Vainshtein, S.I. (1991). *Astrophys. J.* **376**, L21.

Chaboyer, B. & Zahn, J.P. (1992). *Astron. Astrophys.* **253**, 173.

Craddock, G. & Diamond, P.H. (1991). *Phys. Rev. Lett.* **67**, 1535.

Diamond, P.H. & Hahm, T.S. (1995). *Phys. Plasmas* **2**, 3640.

Diamond, P.H., Hughes, D.W. & Kim E.-J. (2005a). In *Fluid Dynamics and Dynamos in Astrophysics and Geophysics*, ed. A.M. Soward, C.A. Jones, D.W. Hughes & N.O. Weiss (CRC Press, London) p. 145.

Diamond, P.H., Itoh, S.-I., Itoh, K. & Hahm, T.S. (2005b). *Plasma Phys. Cont. Fusion* **47**, R35.

Dubrulle, B. & Nazarenko, S. (1997). *Physica D* **110**, 123.

Eddington, A.S. (1926). *The Internal Constitution of the Stars* (Cambridge University Press, Cambridge).

Gilman, P.A. & Fox, P.A. (1997). *Astrophys. J.* **484**, 439.

Gough, D.O. & McIntyre, M.E. (1998). *Nature* **394**, 755.

Gruzinov, A.V. & Diamond, P.H. (1994). *Phys. Rev. Lett.* **72**, 1651.
Gruzinov, I., Das, A. & Diamond, P.H. (2002). *Phys. Let. A* **302**, 119.
Horton, C.W. (1999). *Rev. Mod. Phys.* **71**, 735.
Hughes, D.W. (1991). In *Advances in Solar System Magnetohydrodynamics*, ed. E.R. Priest & A.W. Hood (Cambridge University Press, Cambridge) p. 77.
Hwa, T. & Kardar, M. (1992). *Phys. Rev. A* **45**, 7002.
Itoh, S.-I., Itoh, K., Yoshizawa, A. & Yokoi, N. (2005). *Astrophys. J.* **618**, 1044.
Kim, E.-J. & Dubrulle, B. (2001). *Phys. Plasmas* **8**, 813.
Kim, E.-J., Hahm, T.S. & Diamond, P.H. (2001). *Phys. Plasmas* **8**, 3576.
Kinney, R., McWilliams, J.C. & Tajima, T. (1995). *Phys. Plasmas* **2**, 3623.
Kraichnan, R.H. (1959). *J. Fluid Mech.* **5**, 497.
Kraichnan, R.H. (1965). *Phys. Fluids* **8**, 1385.
Küker, M., Rüdiger, G. & Kitchatinov, L.L. (1993). *Astron. Astrophys.* **279**, 1.
McIntyre, M.E. (2000). In *Perspectives in Fluid Dynamics*, ed. G.K. Batchelor, H.K. Moffatt & M.G. Worster (Cambridge University Press, Cambridge) p. 557.
McIntyre, M.E. (2003). In *Stellar Astrophysical Fluid Dynamics*, ed. M.J. Thompson & J.Christensen-Dalsgaard (Cambridge University Press, Cambridge), p. 111.
Mestel, L. (1999). *Stellar Magnetism* (Clarendon Press, Oxford).
Miesch, M.S. (2001). *Astrophys. J.* **562**, 1058.
Miesch, M. S. (2003). *Astrophys. J.* **586**, 663.
Miesch, M.S. (2005). *Living Rev. Solar Phys.* **2**, 1 (www.livingreviews.org/lrsp-2005-1).
Moffatt, H.K. (1978). *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge).
Naulin, V., Kendl, A., Garcia, O.E., Nielsen, A.H. & Rasmussen, J.J. (2005). *Phys. Plasmas* **12**, 052515.
Nozawa, T. & Yoden, S. (1997). *Phys. Fluids* **9**, 2081.
Orszag, S.A. (1970). *J. Fluid Mech.* **41**, 363.
Parker, E.N. (1966). *Astrophys. J.* **145**, 811.
Parker, E.N. (1993). *Astrophys. J.* **408**, 707.
Pedlosky, J. (1987). *Geophysical Fluid Dynamics* (Springer-Verlag, New York).
Pouquet, A. (1978). *J. Fluid Mech.* **88**, 1.
Pouquet, A, Frisch, U. & Léorat, J. (1976). *J. Fluid Mech.* **77**, 321.
Rhines, P.B. (1975). *J. Fluid Mech.* **69**, 417.
Rhines, P.B. & Young, W.R. (1982). *J. Fluid Mech.* **122**, 347.
Rüdiger, G. & Kitchatinov, L.L. (1977). *Astron. Nachr.* **318**, 273.
Schou, J. & 23 others (1998). *Astrophys. J.* **505**, 390.
Spiegel, E.A. & Zahn, J.P. (1992). *Astron. Astrophys.* **265**, 106.
Sweet, P.A. (1950). *Mon. Not. Roy. Astron. Soc.* **110**, 548.
Tobias, S.M. (2005). In *Fluid Dynamics and Dynamos in Astrophysics and Geophysics*, ed. A.M. Soward, C.A. Jones, D.W. Hughes & N.O. Weiss (CRC Press, London), p. 193.
Tobias, S.M., Brummell, N.H., Clune, T.L. & Toomre, J. (2001). *Astrophys. J.* **549**, 1183.
Vallis, G.K. & Maltrud, M.E. (1993). *J. Phys. Ocean.* **23**, 1346.
Yoshizawa, A., Itoh, S.-I. & Itoh, K. (2003). *Plasma and Fluid Turbulence: Theory and Modelling*, (IoP Publishing, Bristol).
Zeldovich, Ya. B. (1957). *Sov. Phys. J.E.T.P.* **4**, 460.
Zahn, J.P. (1992). *Astron. Astrophys.* **265**, 115.